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# ALGEBRA, CALCULUS & SOLID GEOMETRY

**Bachelor of Arts (B.A.)**

**Three Year Programme**

**New Scheme of Examination**


**DIRECTORATE OF DISTANCE EDUCATION**
**MAHARSHI DAYANAND UNIVERSITY, ROHTAK**
(A State University established under Haryana Act No. XXV of 1975)
NAAC 'A$^{+}$' Grade Accredited University

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**ROHTAK – 124 001**

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**UNIT – 1**

# MATRICES

## 1.0 INTRODUCTION

We have studied about matrices and their properties in the previous classes. Now, we are going to learn about matrices holding some special properties. In this chapter, we learn about symmetric, skew-symmetric, Hermitian, Skew-Hermitian matrices.

We shall also study rank of a matrix, row rank and column rank of a matrix. We shall show that for every matrix its rank, row rank and column rank are all equal.

## 1.1 OBJECTIVES

After going through this unit you will be able to:

- Differentiate between Symmetric and Skew- Symmetric matrices.
- Differentiate between Hermitian and Skeew Herrmitian matrices.
- Know about the sub-matrix and minor of a matrix
- Find the rank of any matrix
- Find the inverse of a matrix
- Differentiate between linearly dependent and independent vectors
- Find characteristics roots and corresponding characteristic vectors of a matrix
- Verify Cayley Hamilton theorem for various matrices and use it to find the inverse of a matrix.
- Learn important theorems related to characteristic roots and characteristics vectors

## 1.2 REVIEW OF MATRICES

### 1.2.1. Matrix

An array of **mn** numbers arranged in **m** rows and **n** columns and bounded by square bracket [ ], brackets ( ) or || || is called **m** by **n** matrix and is represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \dots(1)$$

(1) is known as m × n matrix in which there are m rows and n columns. Each member of m × n matrix is known as an element of the matrix.

**Note**:

1. In general, we denote a Matrix by capital letter $A = [a_{ij}]$, where $a_{ij}$ are elements of Matrix in which its position is in $i^{th}$ row and $j^{th}$ column i.e. first suffix denote row number and second suffix denote column number.
2. The elements $a_{11}$, $a_{22}$,..., $a_{nn}$ in which both suffix are same called diagonal elements, all other elements in which suffix are not same are called non diagonal elements.
3. The line along which the diagonal element lie is called the Principal Diagonal.

**1.2.2. Zero Matrix or Null Matrix.** A Matrix in which each elements is equal to zero is called a zero matrix or null matrix.

e.g., $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

are zero matrix respectively of order $2 \times 3$; $3\times2$ and $3\times3$.

In general we denote zero matrix of order $m\times n$ by $O_{m \times n}$. Matrix other than Zero Matrix are called Non-Zero Matrix.

**1.2.3. Square matrix.** Matrix in which number of row becomes equal to number of column is called square matrix i.e.

If matrix A is of type $m \times n$, where $m = n$ then the matrix is called square matrix otherwise it is called rectangular matrix.

**1.2.4. Row Matrix.** A matrix of type $1 \times n$, having only one row is called row matrix. For example $\begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$ is a row matrix.

**1.2.5. Column Matrix.** A matrix of type $m \times 1$, having only one column is called column matrix. For example $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is a column matrix.

**1.2.6. Diagonal Matrix.** A square matrix in which all non diagonal elements are equal to zero is called diagonal matrix i.e.

A square matrix $A = [a_{ij}]$, is diagonal matrix if $a_{ij} = 0$ for $i \neq j$. Thus

$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ are diagonal matrices.

**1.2.7. Scalar Matrix.** Diagonal matrices in which all diagonal elements are equal are called scalar matrix i.e.

A square matrix $A = [a_{ij}]$, is scalar matrix if $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = k$, for $i = j$ .Thus

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is scalar matrix.

**1.2.8. Unit Matrix or Identity Matrix.** A scalar matrix in which all diagonal elements are unity are called Unit matrix or Identity matrix generally denoted by $I_n$.

A square matrix $A = [a_{ij}]$, is Identity matrix if $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$, for $i = j$ .Thus

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, is identity matrix of order $3 \times 3$.

**1.2.9. Triangular matrix is of two types:**

**(a) Upper Triangular Matrix.** It is a matrix in which all elements below the principal diagonal are zero

e.g. $\begin{pmatrix} 2 & 4 & 8 \\ 0 & -7 & -3 \\ 0 & 0 & 3 \end{pmatrix}$

**(b) Lower Triangular Matrix.** It is a Matrix in which all elements above the principal diagonal are zero

e.g. $\begin{pmatrix} 2 & 0 & 0 \\ 4 & -7 & 0 \\ -6 & 5 & 3 \end{pmatrix}$

**1.2.10. Sub Matrix:** A matrix B obtained by deleting some rows or some column or both of a matrix A, is called a sub matrix of A.

For example. If $A=\begin{pmatrix} 2 & 1 & 5 \\ 0 & -7 & 9 \\ -3 & 4 & 3 \end{pmatrix}$ then the matrices $\begin{pmatrix} 2 & 1 \\ 0 & -7 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 5 \\ 0 & -7 & 9 \end{pmatrix}$ etc. are sub matrix of A.

1.**2.11. Transpose of a matrix**

If matrix A is of type m × n, then the matrix obtained by interchanging the rows and the columns of A is known as Transpose of Matrix A, denoted by $A'$ or $A^T$ i.e.

A= $[a_{ij}]$ of m × n order then

$A'$ or $A^T=[a_{ji}]$ of n × m order Matrix,

Now if A', B' be the transpose of matrix A and B respectively, then

(i) $A = (A')'$ i.e. the transpose of transpose of a matrix A is matrix A itself.

(ii) $(A + B)' = A' + B'$ i.e. the transpose of the sum of two matrices is equal to the sum of their transposes.

(iii) $(kA)'=k A'$, where k is a scalar.

(iv) $(AB)' = B'A'$ i.e. the transpose of the product of two matrices is equal to the product of their transposes, taken in reversed order.

**1.2.12. Conjugate of a Matrix**

If A be a matrix of order m × n, over complex number system, then the matrix obtained from A by replacing each of its elements by their corresponding complex conjugates is called the conjugate of A and is denoted by $\overline{A}$, where $\overline{A}$ is also of same order m × n. If $\overline{A}, \overline{A}$ be the conjugate matrices of A, B respectively, then

(i) $\overline{(\overline{A})} = A$.

(ii) $\overline{(A + B)} = \overline{A} + \overline{B}$, where A and B are conformable for addition.

(iii) $\overline{(kA)} = \overline{k}.\overline{A}$, where k is any complex number.

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(iii) $\overline{(kA)} = \overline{k}.\overline{A}$, where k is any complex number.

(iv) $\overline{AB} = \overline{A}.\overline{B}$, where A and B are conformable for multiplication.

**1.2.13. Transpose Conjugate of a Matrix**

The transpose of the conjugate or conjugate of the transpose of a matrix A is called Transpose conjugate of A and is denoted by $A^{\theta}$. Thus

$A^{\theta} = (\overline{A})' = \overline{(A')}$.

If $A^{\theta}$, $B^{\theta}$ denote the transposed conjugate of A, B respectively, then

(i) $(A^{\theta})^{\theta} = A$.

(ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$, where A and B are conformable for addition.

(iii) $(kA)^{\theta} = \overline{k}.A^{\theta}$, where k is any complex number.

(iv) $(AB)^{\theta} = B^{\theta}.A^{\theta}$, where A and B are conformable for multiplication.

**1.2.14 Adjoint of a Square Matrix**

If A is an n- rowed square matrix, then adjoint of A is defined as a transpose of matrix obtained by replacing each of its elements by its cofactors.

**Theorem 1.1:** If A be an n-square matrix, then $A(\text{adj. }A) = (\text{adj.}A)A = |A| I_n$, where $I_n$ denotes the unit matrix of order n.

**Theorem 1.2:** If A and B are square matrix of the same order n, then $\text{adj. }(AB) = (\text{adj.}B)(\text{adj.}A)$.

**1.2.15 Inverse of Square Matrix**

Let A be n-square matrix, if there exist an n-square matrix B such that

$AB = BA = I_n$, then the matrix A is called invertible and the matrix B is called inverse of A. Inverse of a square matrix is denoted by $A^{-1}$.

Note. 1. From definition it is clear that A is the inverse of B.

2. A non-square matrix does not have any inverse.

**1.2.16 Singular and Non Singular Matrices**

A square matrix A is said to be singular or non singular according as $|A| = 0$ or $|A| \neq 0$.

**Theorem 1.3:** If A and B are two non singular matrix of order n, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Given, A and B are two non singular matrices.

$\therefore |A \neq 0|$ and $|B| \neq 0$ and hence $|AB| = |A||B| \neq 0$

Consider

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} && [\because BB^{-1} = I_n] \\
&= AI_nA^{-1} \\
&= (AI_n)A^{-1} = AA^{-1} && [\because AI_n = A] \\
&= I_n && [\because AA^{-1} = I_n]
\end{aligned}$$

Now consider

$$(B^{-1}A^{-1})(AB)=B(A^{-1}A)B^{-1} \qquad [\because AA^{-1}=I_n]$$
$$= BI_nB^{-1}$$
$$= (BI_n)B^{-1}=BB^{-1} \qquad [\because BI_n=B]$$
$$= I_n \qquad [\because BB^{-1}=I_n]$$

So, $(AB)(B^{-1}A^{-1})=(B^{-1}A^{-1})(AB)=I_n \Rightarrow (AB)^{-1}=B^{-1}A^{-1}$.

**Theorem 1.4:** The adjoint of non singular matrix is non singular.

Proof: Let A be a non singular matrix of order n. Then $|A| \neq 0$.

As we know that $A(\text{adj. }A)=|A|I_n$

Now, taking determinant both sides, we get

$|A||\text{adj.}A|=|A|^n$, dividing both side by $|A| \neq 0$, we get

$|\text{adj.}A|=|A|^{n-1}$ as $|A| \neq 0 \Rightarrow |\text{adj.}A| \neq 0$

Hence adjoint of a non singular matrix is non singular.

**Theorem 1.5:** If A is a non singular matrix of order n. then

(a) $|\text{adj.}A|=|A|^{n-1}$ (b) $\text{adj.}(\text{adj. }A)=|A|^{n-2}A$

**Theorem 1.6:** The inverse of every square matrix, if exist, is unique.

**Theorem 1.7:** The necessary and sufficient condition for any square matrix A to be invertible is that A is non singular.

**Theorem 1.8:** If A is non singular matrix, then $\det(A^{-1})=(\det A)^{-1}$.

**Theorem 1.9:** If A is non singular matrix, then $A'$ is also non singular and $(A')^{-1}=(A^{-1})'$.

**1.2.17. Solution of System of Linear Equations**

Any given system of linear equations may be written in term of matri*x*, such that

$$AX = B \qquad \ldots(i)$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A is known as co-efficient matri*x*.

If we multiply both sides of (i) by the reciprocal matri*x* $A^{-1}$, then we get $A^{-1}AX = A^{-1}B$

$$(A^{-1}A)X = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ where } \Delta \neq 0$$

$$= \frac{1}{\Delta}\begin{bmatrix} A_1d_1 + A_2d_2 + A_3d_3 \\ B_1d_1 + B_2d_2 + B_3d_3 \\ C_1d_1 + C_2d_2 + C_3d_3 \end{bmatrix} \qquad \ldots(ii)$$

Hence from (ii) equating the values of $x$, y and z we get the desired result.

This method is true only when (i) $\Delta \neq 0$ (ii) number of equations and number of unknowns (e.g. $x$, y, z etc.) are the same.

***Example 1. Solve the equations with the help of determinants :***

$$x + y + z = 3, \quad x + 2y + 3z = 4, \quad x + 4y + 9z = 6.$$

***Sol.*** The co-efficient determinant is $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2 \neq 0$

$$\therefore \qquad x = \frac{1}{2}\begin{bmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{bmatrix} \Rightarrow \qquad x = \frac{1}{2} \times 4 = 2$$

$$y = \frac{1}{2}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{bmatrix} \Rightarrow \qquad y = \frac{1}{2}(2) = 1 \Rightarrow y = 1$$

$$z = \frac{1}{2}\begin{bmatrix} 1 & 1 & +3 \\ 1 & 2 & +4 \\ 1 & 4 & +6 \end{bmatrix} \Rightarrow z = \frac{1}{2}[-4 + 6 + (4 - 6)] = 0 \Rightarrow \qquad z = 0$$

$\therefore$ Solution is $x = 2$, y = 1, z = 0.

## 1. 3. Symmetric And Skew Symmetric Matrices

### 1.3.1. Symmetric Matrix

A square matrix A = $[a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j

Examples $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 7 \end{bmatrix}$

### 1.3.2. Skew Symmetric Matrix

If a square matrix A has its elements such that $a_{ij} = -a_{ji}$ for i and j and the leading diagonal elements are zeros, then matrix A is known as skew matrix. For example $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & g \\ -h & 0 & -f \\ -g & f & 0 \end{bmatrix}$ are skew symmetric matrices.

**Example 1:** Every square matrix can be expressed as the sum of symmetric matrix and a skew-symmetric matrix in one and only one way.

***Solution.*** If A be any square matrix, then we consider

$$B = \frac{1}{2}(A + A') \qquad \text{and} \qquad C = \frac{1}{2}(A - A')$$

$$\Rightarrow \qquad B + C = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(A + A' + A - A') = A$$

Similarly $\quad B' = \frac{1}{2}(A + A')' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}[A' + A] = \frac{1}{2}(A + A') = B$

i.e. $B' = B$. This implies B is symmetric matrix

Now, we consider

$$C' = \frac{1}{2}(A - A')' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A)$$

$$-\frac{1}{2}(A - A') = -C.$$

i.e. $C' = -C$.

Hence C is a skew symmetric matrix.

Therefore, every square matrix A can be expressed as A = B + C, where B = $\frac{1}{2}$ (A + A′), which is symmetric matrix and C = $\frac{1}{2}$(A − A′) is skew skew-symmetric matrix ($\because$ C′ = −C). The same process can be used to show that the result is unique.

**Example 2:** Every skew symmetric matrix of odd order is singular.

Proof: Let A be a Skew Symmetric matrix of order n, where n is odd.

$\therefore \qquad A' = -A$

Taking determinant both sides

So,

$$|A'| = |-A| = |(-1)A|$$

$$|A| = (-1)^n |A| \qquad (\because |kA| \text{ where k is scalar} = k^n |A|)$$

$$|A| = -|A| \qquad (\because \text{ n is odd})$$

$$2|A| = 0 \Rightarrow |A| = 0$$

Thus, A is a singular matrix.

## 1.4 HERMITIAN AND SKEW HERMITIAN MATRIX

### 1.4.1. Hermitian Matrix

A square matrix $A=[a_{ij}]$ over the complex numbers is said to be Hermitian if the transposed conjugate of the matrix is equal to the matrix itself i.e. $A^\theta = A$.

Suppose $A=[a_{ij}]$ is of the type m × n, then $A^\theta=[a_{ij}]$ will be of the type n × m where $a_{ij} = \overline{a_{ji}}$

So, for the matrix A to be Hermitian , m=n and $a_{ij} = \overline{a_{ji}}$ for all i and j.

For example $\begin{bmatrix} 0 & 2-3i \\ 2+3i & 1 \end{bmatrix}, \begin{bmatrix} 10 & 1+i & i \\ 1-i & 8 & 5+4i \\ -i & 5-4i & 0 \end{bmatrix}$ are Hermitian matrices.

**Corollary:** A Hermitian matrix has all its diagonal elements as real numbers.

**Proof**: Let A be Hermitian matrix.

$\therefore a_{ij} = \overline{a_{ji}}$ , for all i and j.

Putting j = I for the diagonal elements, we have

$a_{ij} = \overline{a_{ji}}$ for all i

$\Rightarrow \alpha + i\beta = \alpha - i\beta \qquad [\because a_{ij} = \alpha + i\beta \Rightarrow \overline{a_{ij}} = \alpha - i\beta]$

$\Rightarrow 2i\beta = 0 \Rightarrow \beta = 0$

$\therefore a_{ij} = \alpha$ , Thus the diagonal elements of a Hermitian metrix are real numbers.

**1.4.2. Skew Hermitian Matrix**

A square matrix $A=[a_{ij}]$ over the complex numbers is said to be Skew Hermitian if the transposed conjugate of the matrix is equal to the negative of matrix itself i.e. $A^{\theta}=-A$ .

Suppose $A=[a_{ij}]$ is of the type m × n, then $A^{\theta}=[a_{ij}]$ will be of the type n × m where $a_{ij} = -\overline{a_{ji}}$

So, for the matrix A to be Skew Hermitian , m=n and $a_{ij} = -\overline{a_{ji}}$ for all i and j.

For example $\begin{bmatrix} 0 & 2-3i \\ -2+3i & 0 \end{bmatrix}, \begin{bmatrix} 10i & 1+i & i \\ -1+i & 8i & 5+4i \\ i & -5+4i & 0 \end{bmatrix}$ are Skew Hermitian matrices.

**Corollary:** A Skew Hermitian matrix has all its diagonal elements as either zero or purely imaginary.

Proof: Let A be Skew Hermitian matrix.

$\therefore a_{ij} = -\overline{a_{ji}}$ , for all i and j.

Putting j = i for the diagonal elements, we have

$a_{ij} = -\overline{a_{ji}}$ for all i

$\Rightarrow \alpha + i\beta = -\alpha + i\beta \qquad [\because a_{ij} = \alpha + i\beta \Rightarrow -\overline{a_{ij}} = -\alpha + i\beta]$

$\Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

$\therefore a_{ij} = i\beta$ , Thus the diagonal elements of a Skew Hermitian metrix are either zero or purely imaginary

**Example 1:** If A is square matrix then prove that

(i) . $A+A^{\theta}$ is Hermitian matrix.

(ii) $A-A^{\theta}$ is skew Hermitian matrix.

Solution: (i) Consider

$(A+A^{\theta})^{\theta}=A^{\theta}+(A^{\theta})^{\theta}$

$= A^{\theta}+A \qquad [Q(A^{\theta})^{\theta}=A]$

$=A+A^{\theta}$

Thus, $A+A^{\theta}$ is Hermitian matrix.

(ii) Consider

$(A-A^{\theta})^{\theta}=A^{\theta}-(A^{\theta})^{\theta}$

$= A^{\theta}-A \qquad [\because (A^{\theta})^{\theta}=A]$

$=-(A-A^{\theta})$

Thus, $A-A^{\theta}$ is Skew Hermitian matrix.

**Example 2:** Every square matrix A can be expressed in one and only one way as $P+iQ$, where P and Q are Hermitian matrices.

**Solution**. We have

$$A=\frac{1}{2}(2A)=\frac{1}{2}[A+A^{\theta}+A-A^{\theta}]$$

$$=\frac{1}{2}(A+A^{\theta})+\frac{1}{2}(A-A^{\theta})$$

$$=\frac{1}{2}(A+A^{\theta})+i.\frac{1}{2i}(A-A^{\theta})=P+iQ$$

where, $P=\frac{1}{2}(A+A^{\theta})$ and $Q=\frac{1}{2i}(A-A^{\theta})$

Now, $P^{\theta}=\frac{1}{2}(A+A^{\theta})^{\theta}=\frac{1}{2}(A^{\theta}+A)=P$

and

$$Q^{\theta}=-\frac{1}{2i}(A-A^{\theta})^{\theta}=-\frac{1}{2i}(A^{\theta}-A)$$

$$=\frac{1}{2i}(A-A^{\theta})=Q \qquad [\because (kA)^{\theta}=\bar{k}A^{\theta}]$$

Thus both P and Q are Hermitian.

Hence, A can be expressed as $P+iQ$, where P and Q are Hermitian matrices.

To prove that this expression of A is unique:

Let, if possible $A=R+iS$ be another expression for A where R and S are Hermitian. We shall prove that R=P and S=Q

Now, $A^{\theta}=(R+iS)^{\theta}=R^{\theta}+\bar{i}S^{\theta}=R-iS$ $\quad [\because$ R and S are Hermitian$]$

$$A+A^{\theta}=(R+iS)+(R-iS)=2R$$

$$\Rightarrow R=\frac{1}{2}(A+A^{\theta})=P$$

Also

$$A-A^{\theta}=(R+iS)-(R-iS)=2iS$$

$$\Rightarrow S=\frac{1}{2i}(A-A^{\theta})=Q$$

Hence, the expression for A is unique.

**Check Your Progress**

**1.** Express the following matrix as the sum of a symmetric and skew symmetric matrix

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}.$$

Ans. Symmetric matrix is $\begin{pmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{pmatrix}$ and skew symmetric matrix is $\begin{pmatrix} 0 & \frac{5}{2} & -2 \\ -\frac{5}{2} & 0 & 2 \\ 2 & -2 & 5 \end{pmatrix}$.

2. Show that value of determinant of skew symmetric matrix of odd order is always zero.

3. If A is any square matrix, prove that AA' and A'A are both symmetric

4. If A is a skew symmetric matrix of order n, then show that adj.A is symmetric or skew symmetric according as n is odd or even .

## 1.5. RANK OF A MATRIX

Let A be m × n matrix. So, A has sub-matrices of various orders. The determinant of any such sub-matrices is defined as minor of matrix A of order r where r < m and r < n [or r ≤ m, m ≤ n]. If all minors of order (r + 1) are zeros and we have at least one non-zero minor of order r, then it is said that the rank of matrix A is r and rank of r is represented by ρ(A) = rank of A = r.

Thus from the above definition of the rank of a matrix A, we have the following observations:

(a) If A = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = null matrix then rank of A = ρ(A) = 0.

(b) If A is a nonzero matrix then rank of A i.e. □ (A)≥ 1.

(c) If A is m × m unit matrix then | A | = $\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$ = 1 ≠ 0

i.e. rank of A = ρ(A) = m.

(d) If A is m × n matrices, then ρ(A) ≤ min of m and n.

(e) If all minors of order r are equal to zero then rank of A = ρ(A) < r.

**Example 1. Determine the rank of the matrix A =** $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$**.**

**Sol.** Operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & -5 & 7 \end{bmatrix}$$

Again operating $R_2 \to -\frac{1}{2}R_2$, we get

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix}$$

Next, operating $R_3 \to R_3 + 5R_2$, we get

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix}$$

This implies that rank of A = 3.

**Example 2. Determine the rank of the matrix A =** $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$**.**

**Sol.** Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Next, operating $R_3 \to R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$ Here 3rd order minor is zero.

But 2nd order minors exist i.e. $\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2 - 6 = -8 \neq 0$

So, the rank of matrix A = 2.

**Check Your Progress**

5. If A is an n-square matrix of rank n-1, show that adj. A ≠ O.
6. If A is non zero column matrix and B is non zero row matrix, show that $\rho(AB)=1$

### 1.5.1. Elementary Transformations (or Operations) on A Matrix

The following operations on a matrix are called elementary transformations (i.e. E operations or E-transformation)

(a) The interchange of $i^{th}$ and $j^{th}$ rows is represented by $R_{ij}$, and the interchange of $i^{th}$ and $j^{th}$ columns is represented by $C_{ij}$.

(b) Multiplication of each element of a row or a column by a non-zero number k.

i.e. the multiplication of $i^{th}$ row by k is represented by $kR_i$ and the multiplication of $i^{th}$ column by p is represented by $pC_i$.

(c) Addition of m times the elements of a row (or column) to the corresponding elements of another row (or column) multiplied by n, where $m \neq 0$, $n \neq 0$.

The addition of m times $i^{th}$ row to the n times $j^{th}$ row is represented by $mR_i + nR_j$.

If a matrix B is obtained from matrix A by such transformation, then the matrix B is called the **equivalent matrix** to matrix A. If matrix B is obtained from A by applying finite number of elementary row operation on A, then B is **row equivalent** to A. If matrix B is obtained from A by applying finite number of elementary column operation on A, then B is **column equivalent** to A.
Two such equivalent matrices A and B are denoted as A ~ B and the symbol ~ is used for equivalence. So, two equivalent matrices have the same order and same rank.

**Theorem 1.15:** The rank of a matrix remains unaltered by the application of elementary row and column operations.

**Proof:** Let A be m × n matrix, such that

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

If $\rho(A)=r$, then every minor of order r+1, if any vanishes and there will be at least one non zero minor of A of order r.

Now consider a minor of order r+1 denoted by $M_{r+1}$.

(1) If we interchange any two rows or columns of A, the value of determinant remains unaltered by numerically value but the sign is changed.

(2) If one row and column of A is multiplied by any scalar k, the value of determinant multiplied by the scalar k.

(3) If we apply $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$, then the determinant remains unchanged.

We have seen that in each case of elementary row/column operation, the value of $M_{r+1}$ remains unaltered. Since all minor of order r+1 in A are zero , they will also be equal to zero in all equivalent matrices.

Thus $\rho(B) \le \rho(A)$, where B is a matrix obtained by elementary operations.

Again, A can be obtained back from B by elementary operations of the same type and so, we have $\rho(A) \le \rho(B)$

Hence we conclude that $\rho(A)=\rho(B)$ i.e. the rank of any matrices remains unaltered by the application of finite chain of row/ column operations.

**1.5.2 Row Echelon Matrix:**

A matrix $A=[a_{ij}]$ is called a row echelon matrix if the following conditions are satisfied:

(1) The first non zero element in each non zero row is unity which is called leading entry of row.

(2) All the non zero rows,, precede the zero rows, if any.

(3) The number of zeros before the leading entry in each row is less than the number of such zero's in the succeeding rows.

For example $A=\begin{bmatrix} 1 & 2 & 4 & 5 & 1 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form.

### 1.5.3 Row reduced Echelon Form:

In addition to the above three conditions, if a matrix satisfies the following conditions:

Each column which contains a leading entry of a row has all other entries zeros, then the matrix is said to be in row reduced echelon matrix.

### 1.5.4 Row Rank and Column Rank of a Matrix

Row rank of a matrix, say A is the number of non zero rows in the row echelon matrix A and is denoted by $\rho_R(A)$.

Column Rank of a matrix, say A is the number of non zero columns in the column echelon matrix A and is denoted by $\rho_C(A)$.

Note: (i) Every matrix is row equivalent to row echelon matrix.

(ii) Every matrix is column equivalent to a column echelon matrix.

(iii) If a matrix A is in row echelon form, then its transpose is in column echelon form.

**Example. 1.10:** Reduce the matrix $A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 & 3 \\ 0 & 2 & 6 & 1 & 3 & 9 \\ 0 & 4 & 12 & -2 & 10 & 7 \end{bmatrix}$ to the row reduced echelon form and hence find its rank.

Solution: Applying $R_2 \to R_2\text{-}R_1, R_3 \to R_3\text{-}2R_1$, and $R_4 \to R_4\text{-}4R_1$ on the matrix A,

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 & 7 \\ 0 & 0 & 0 & 2 & -2 & 3 \end{bmatrix}$$

Applying $R_1 \to R_1\text{+}R_2, R_3 \to R_3\text{-}3R_2$, and $R_4 \to R_4\text{-}2R_2$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Applying $R_1 \to R_1\text{-}3R_3, R_2 \to R_2\text{-}2R_3$, and $R_4 \to R_4\text{+}R_3$

$$A=\begin{bmatrix} 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the required row reduced echelon form of the matrix A. Since, the number of non zero rows is 3, thus row rank of A is 3.

**Check Your Progress**

**7.** Find the row rank of matrix

$$A=\begin{bmatrix} 3 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 \\ 7 & 6 & 2 & 5 \end{bmatrix}.$$

**Ans**. 3

8. Reduce the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ to the row reduced echelon form. Also find their row rank.

**Ans**. 2

### 1.5.5 Rank of Product of Two Matrices

**Theorem 1.16:** The rank of product of two matrices cannot exceeds the rank of either matrix i.e. $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$

Proof: Let $\rho(A)=r_1, \rho(B)=r_2$ and $\rho(AB)=r$

Now we reduce the matrix A to the normal form, $A \sim \begin{bmatrix} M \\ O \end{bmatrix}$

where M is a matrix of $r_1$ rows and $r_1$ rank

Post multiplying by B, we get $AB \sim \begin{bmatrix} M \\ O \end{bmatrix} B$

The matrix $\begin{bmatrix} M \\ O \end{bmatrix} B$ will have at most first $r_1$ non zero rows which can obtained by multiplying first $r_1$ non zero rows of M with the column of B.

Thus, $\rho(AB) \sim \rho\left[\begin{bmatrix} M \\ O \end{bmatrix} B\right] \le r_1$

$$\Rightarrow \rho(AB) \le r_1 \Rightarrow \rho(AB) \le \rho(A)$$

Thus, the rank of product AB $\le$ rank of of the prefactor A ......(i)

We have, $r=\rho(AB)=\rho[(AB)']$

[As the rank of transpose of a matrix is same as that of original matrix]

$=\rho(B'A') \le \rho(B')$ [ Rank of the product $\le$ Rank of prefactor]

$\therefore \rho(AB) \le \rho(B)$.

**Theorem 1.17** The rank, column rank and row rank of a matrix are all equal.

**Proof**: Let r be rank, s be the row rank and t be the column rank of a matrix A of type m × n.

i.e. $\rho_R(A)=s$, $\rho_C(A)=t$ and $\rho(A)=r$

As the row rank of the matrix A is s, thus there must be non singular matrix P such that $PA=\begin{bmatrix} B \\ O \end{bmatrix}$, where B is a matrix of type s × n. We know that pre or post multiplication by non singular matrix does not affect the rank of matrix, therefore

$\rho(PA)=\rho(A)=r$ .......(i)

Since every square minor of of PA of order (s+1) has atleast one row of zeros, thus, each minor of PA of order (s+1) is equal to zero.

$\rho(PA) \le s \Rightarrow r \le s$ [Using (i)] ...(ii)

As $\rho(PA)=r$, thus there must be a non singular matrix Q such that $QA=\begin{bmatrix} C \\ O \end{bmatrix}$, where C is a matrix of type r × n.

$\therefore \rho_R(QA)=\rho_R(A)=s$

Since the matrix QA has only r non zero rows, thus

$\rho(QA) \le r \Rightarrow s \le r$ ...(iii)

From (ii) and (iii), we have r = s ...(iv)

Similarly, r=t and hence s=r=t

Thus, $\rho(A)=\rho_R(A)=\rho_C(A)$.

**Example 1:** Express $A=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -2 \\ 1 & 2 & 1 \end{bmatrix}$ as the product of elementary matrices.

**Solution:** Applying $R_1 \to R_2\text{-}2R_1, R_3 \to R_3+(-1)R_1$ on the matrix A,

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Applying $C_2 \to C_2+(-2)C_1, C_3 \to C_3+(1)C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Applying $R_3 \to R_3\left(\frac{1}{2}\right)$,

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we observe that by performing the elementary operations

$R_1 \to R_2\text{-}2R_1, R_3+(-1)R_1, C_2+(-2)C_1, C_3+(1)C_1, R_3\left(\frac{1}{2}\right)$

Successively, we can reduce matrix A to $I_3$

If $RE_{21}(-2), RE_{31}(-1), CE_{21}(-2), CE_{31}(1), RE_3\left(\frac{1}{2}\right)$ are the corresponding elementary matrices, then

$$I_3 = RE_3\left(\frac{1}{2}\right)RE_{31}(-1)RE_{21}(-2)CE_{21}(-2)CE_{31}(1)$$

$$\Rightarrow A = [RE_{21}(-2)]^{-1}[RE_{31}(-1)]^{-1}\left[RE_3\left(\frac{1}{2}\right)\right]^{-1} I_3[CE_{21}(-2)]^{-1}[CE_{31}(1)]^{-1}$$

$$\Rightarrow A = [RE_{21}(2)][RE_{31}(1)]\left[RE_3(2)\right][CE_{31}(-1)][CE_{21}(2)]$$

$$\Rightarrow A = \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$$

## 1.6 ELEMENTARY MATRICES

Elementary matrix is a matrix which is obtained from an identity matrix $I_n$ by a single elementary transformation. For example, consider

$$I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Applying $R_1 \to R_1 + R_2 + 3R_3$

$$I_3 \sim \begin{bmatrix}1+0+3.0 & 0+1+3.0 & 0+0+0\\0&1&0\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&1&3\\0&1&0\\0&0&1\end{bmatrix}$$

which is an elementary matrix.

**Note:** All the elementary matrices are non singular.

### 1.6.1 Some Theorems on Elementary Matrices

**Theorem 1.18:** If A and B Are two matrices over the field F of the type (m×n) and (n×p) respectively, then application of any elementary row (column) operation to A(B) results in the application of the same to the product matrix AB and vice versa.

Proof: Let A= $[a_{ij}]_{m\times n}$ and B= $[b_{ij}]_{n\times p}$

i.e. $A = \begin{bmatrix}R_1\\R_2\\:\\R_m\end{bmatrix}$ and $B = \begin{bmatrix}C_1 & C_2 & .. & C_p\end{bmatrix}$

where $R_i = \begin{bmatrix}a_{i1} & a_{12} & .. & a_{1n}\end{bmatrix}, 1 \le i \le m$

and $C_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}, 1 \le j \le p$.

In other words $R_1, R_2, \ldots\ldots, R_m$ are rows of A and $C_1, C_2, \ldots\ldots, C_p$ are columns of B.

$$AB = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_s \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} C_1 & C_2 & .. & C_p \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & \ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots & R_2C_p \\ \vdots & \vdots & \ldots & \vdots \\ R_iC_1 & R_iC_2 & \ldots & R_iC_p \\ \vdots & \vdots & \ldots & \vdots \\ R_sC_1 & R_sC_2 & \ldots & R_sC_p \\ \vdots & \vdots & \ldots & \vdots \\ R_mC_1 & R_mC_2 & \ldots & R_mC_p \end{bmatrix}$$

where $R_iC_j = \begin{bmatrix} a_{i1} & a_{12} & .. & a_{1n} \end{bmatrix} . \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}; 1 \le i \le m, 1 \le j \le p$

$$= a_{i1}b_{1j} \quad a_{i2}b_{2j} \quad .. \quad a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Case I Applying elementary row operation $R_i + tR_s$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i + tR_s \\ \vdots \\ R_s \\ \vdots \\ R_m \end{bmatrix}$$

Then $\text{AB}=\begin{bmatrix} R_1C_1 & R_1C_2 & \ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots & R_2C_p \\ : & : & \ldots & : \\ (R_i+tR_s)C_1 & (R_i+tR_s)C_2 & \ldots & (R_i+tR_s)C_p \\ : & : & \ldots & : \\ R_sC_1 & R_sC_2 & \ldots & R_sC_p \\ : & : & \ldots & : \\ R_mC_1 & R_mC_2 & \ldots & R_mC_p \end{bmatrix}$

Case II Applying $tR_i, t \neq 0$ to AB, we get

$$\text{AB}=\begin{bmatrix} R_1C_1 & R_1C_2 & \ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots & R_2C_p \\ : & : & \ldots & : \\ tR_iC_1 & tR_iC_2 & \ldots & tR_iC_p \\ : & : & \ldots & : \\ R_sC_1 & R_sC_2 & \ldots & R_sC_p \\ : & : & \ldots & : \\ R_mC_1 & R_mC_2 & \ldots & R_mC_p \end{bmatrix}=\begin{bmatrix} \text{R}_1 \\ \text{R}_2 \\ : \\ tR_i \\ : \\ \text{R}_\text{s} \\ : \\ \text{R}_\text{m} \end{bmatrix}\begin{bmatrix} \text{C}_1 & \text{C}_2 & .. & C_p \end{bmatrix}$$

The other cases can similarly be dealt with. Further, on similar lines it can be verified that application of elementary column operation on B results in the application of the same on AB and vice-versa.

**Theorem 1.19:** Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication (post-multiplication) with corresponding elementary matrix.

**Proof:** Let A be m×n matrix.

We write $\text{A}=\text{I}_\text{m}\text{A}$

If α denotes any elementary row transformation, we have

$$\alpha\text{A}=\alpha(\text{I}_\text{m}\text{A})=(\alpha\text{I}_\text{m})\text{A}=\text{EA},$$

Where E is the elementary matrix corresponding to the same row transformation $\alpha$ .

Similarly,we write $\text{A}=\text{AI}_\text{n}$

If $\alpha$ denotes any elementary column transformation, we get

$\alpha(\text{A})=\alpha(\text{AI}_\text{n})=\text{A}(\alpha\text{I}_\text{n})=\text{AE}_1$

where $\text{E}_1$ is the elementary matrix corresponding to the same column transformation $\alpha$ .

**Theorem 1.20:** The inverse of an elementary matrix is an elementary matrix of the same type:

(i) $(RE_{ij})^{-1}=(RE_{ij})$ (ii) $[\text{RE}_\text{i}(\text{k})]^{-1}=\text{RE}_\text{i}(\text{k}^{-1})$ (iii) $[\text{RE}_\text{ij}(\text{k})]^{-1}=\text{RE}_\text{ij}(\text{-k})$

(iv) $[\text{CE}_\text{ij}(\text{k})]^{-1}=\text{CE}_\text{ij}(\text{-k})$ .

**Proof:** (i) $RE_{ij}$ has been obtained from I by applying $R_{i,j}$ and obviously, if we apply $R_{i,j}$ again we get I. But applying $R_{i,j}$ to $RE_{ij}$ means pre-multiplication of $RE_{ij}$ with corresponding elementary matrix $RE_{ij}$, i.e.

$$(RE_{ij})(RE_{ij}) = I$$

$$\therefore \quad (RE_{ij})^{-1} = RE_{ij}$$

(ii) On applying $R_i(k)$ on I, we get $RE_i(k), k \neq 0$ and if we apply $R_i(k^{-1})$ on $RE_i(k)$, we get I. Similar to (i), we have $[RE_i(k^{-1})][RE_i(k)]=I$

$$\Rightarrow [RE_i(k)]^{-1} = [RE_i(k^{-1})]$$

Similarly, we can prove (iii) and (vi).

**Theorem 1.21:** The rank of the transpose of a matrix is equal to the original matrix i.e. $\rho(A^T)=\rho(A)$

**Proof:** Consider a matrix A of order ( $m\times n$ ).Then the transpose of A denoted by $A^T$ will be of order $(n\times m)$.

Let $\rho(A)=r$ i.e. there exist a non zero monor of order r of $|A|$.

Let $|M_r|$ be such minor.

Then $M_r^T$ which is transpose of $M_r$, will be sub-matrix of $A^T$.

Since the value of determinant remains unchanged when rows and columns are interchanged, therefore

$$|M_r| = |M_r^T|$$

Since $\qquad |M_r| \neq 0 \quad \Rightarrow |M_r^T| \neq 0$

$$\Rightarrow \qquad \rho(A^T) \geq r \qquad \qquad ....(1)$$

Now consider a square sub-matrix $N^T$ of order (r+1) from $A^T$. Then N is a sub-matrix of A. Since $\rho(A)=r$, therefore all minors of order (r+1) will be zero.

$$\Rightarrow \qquad |N| = 0$$

$$\Rightarrow \qquad |N^T| = 0$$

$$\Rightarrow \qquad \rho(A^T) \leq r$$

From (1) and (2), $\qquad \rho(A^T)=r$

Hence $\qquad \rho(A^T)=\rho(A)$.

**Theorem 1.22:** If A be an $m\times n$ matrix of rank r, there exist non singular matrix P and Q such that

$$PAQ=\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Proof: As A is an $m\times n$ matrix of rank r, therefore it can be transformed into the form $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by elementary transformations.

Since elementary row (column) operations are equivalent to pre (post) multiplication by the corresponding elementary matrices, therefore there exist elementary matrices $P_1, P_2, .....P_k$ and $Q_1, Q_2, .....Q_k$ such that

$$P_k P_{k-1}, .....P_2 P_1 A Q_1 Q_2 .....Q_s = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

Now, since each elementary matrix is non singular therefore product $P=P_k P_{k-1}.....P_1$ and $Q=Q_1, Q_2, .....Q_s$ are non singular matrices, such that $PAQ=\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

**Theorem 1.23:** If A be an $m\times n$ matrix of rank r, then there exist non singular matrices P such that $PA=\begin{bmatrix} B \\ O \end{bmatrix}$, where B is an $r\times n$ matrix of rank r and O is $(m\text{-}r)\times n$ matrix.

Proof: We know that if A is $m\times n$ matrix of rank r, therefore there exist matrix P and Q such that $PAQ=\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$.

Also every non singular matrix can be expressed as product of elementary matrices. Since Q be non singular matrices means $Q^{-1}$ exists and can be expressed as product of elementary matrices $K_1 K_2 K_3 ... K_t$.

$\therefore$ Equation (i) becomes $PA=\begin{bmatrix} I_r & O \\ O & O \end{bmatrix} Q^{-1} = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} K_1 K_2 K_3 ... K_t$, since post multiplication of elementary matrices amounts to E—column transformations, last m-r rows of (1) being zero rows, remain zero rows on applying $K_1 K_2 K_3 ... K_t$.

Thus, We get a relation of the form $PA=\begin{bmatrix} B \\ O \end{bmatrix}$, where B is an $r\times n$ matrix.

Since elementary transformations do not alter the rank, therefore the rank of matrix $\begin{bmatrix} B \\ O \end{bmatrix}$ is r. Since B has r rows, so $\rho(B)=r$ and last m-r rows of $\begin{bmatrix} B \\ O \end{bmatrix}$ are zero rows.

**Theorem 1.24:** If A be an $m\times n$ matrix of rank r, then there exist non singular matrices Q such that $AQ=\begin{bmatrix} C & O \end{bmatrix}$, where C is an $m\times r$ matrix of rank r and O is $m\times(n\text{-}r)$ matrix.

Proof: Do yourself as above theorem.

**Check Your Progress**

1. Find the rank of matrix $\begin{pmatrix} 1 & 2 & -1 & 3 \\ -2 & -4 & 4 & -7 \\ 1 & 2 & 1 & 2 \end{pmatrix}$ by reducing it to normal form.

Ans. Rank = 2.

2. Express $A=\begin{pmatrix}1 & 2 & -1\\ 2 & 5 & -2\\ 1 & 2 & 1\end{pmatrix}$ as the product of elementary matrices.

Ans. $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

**1.7 Inverse of Matrix**

If A is a non singular matrix, then inverse of matrix A exist and is defined as matrix $A^{-1}$ satisfies $AA^{-1}=A^{-1}A=I$, where I is unit matrix of same order as that of the matrix A. To find the inverse of matrix A write $A=IA$, then perform same suitable elementary row (column) operations on the matrix A and on the right hand side till we reach the result $I=BA$. Then $A^{-1}=B$.

**Example 1:** Find the inverse of matrix $A=\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 1\\ 5 & 2 & 3\end{bmatrix}$ using the elementary operations.

**Solution.** We write $A=IA$ i.e., $\begin{bmatrix}1 & 3 & 2\\ 0 & 4 & 1\\ 5 & 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$

Operating $R_3 \to R_3+(-5)R_1, R_2 \to R_2 \times \frac{1}{4}$

we get, $\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & \frac{1}{4}\\ 0 & -13 & -7\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} & 0\\ -5 & 0 & 1\end{bmatrix}A$

Operating $R_1 \to R_1+(-3)R_2, R_3 \to R_3+13R_2$,

$$\begin{bmatrix}1 & 0 & \frac{5}{4}\\ 0 & 1 & \frac{1}{4}\\ 0 & 0 & -\frac{15}{4}\end{bmatrix}=\begin{bmatrix}1 & -\frac{3}{4} & 0\\ 0 & \frac{1}{4} & 0\\ -5 & \frac{13}{4} & 1\end{bmatrix}A$$

Operating $R_3 \to R_3 \times \left(\frac{-4}{15}\right), R_1 \to R_1+\left(\frac{-5}{4}\right)R_3, R_2 \to R_2+\left(\frac{-1}{4}\right)R_3$,

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{2}{3} & \frac{1}{3} & \frac{1}{3}\\-\frac{1}{3} & \frac{7}{15} & \frac{1}{15}\\\frac{4}{3} & -\frac{13}{15} & -\frac{4}{15}\end{bmatrix}A=\frac{1}{15}\begin{bmatrix}-10 & 5 & 5\\-5 & 7 & 1\\20 & -13 & -4\end{bmatrix}A$$

$$A^{-1}=\frac{1}{15}\begin{bmatrix}-10 & 5 & 5\\-5 & 7 & 1\\20 & -13 & -4\end{bmatrix}.$$

**Problems to Check The Progrress**

**9.** Using elementary operation, find the inverse of the following matrices.

$$A=\begin{pmatrix}1 & 2 & -1\\-1 & 1 & 2\\2 & -1 & 1\end{pmatrix}$$

Ans. $A^{-1}=\frac{1}{14}\begin{pmatrix}3 & -1 & 5\\5 & 3 & -1\\-1 & 5 & 3\end{pmatrix}.$

## 1.8 LINEAR DEPENDENCE AND INDEPENDENCE OF ROW & COLUMN MATRICES.

Any quantity having n components is called a vector of order n. If $a_1, a_2, .....a_n$ are elements of fields (F, +, .), then these numbers written in a particular order form a vector.

Thus an n-dimensional vector X over a field (F,+, .) is written as X=$(a_1, a_2, .....a_n)$ where $a_i \in F$.

Row matrix of type $1\times n$ is n—dimensional vector written as $X=[a_1,a_2,.....a_n]$

Column matrix of type $n\times 1$ is also n dimensional vector written as

$$X=\begin{bmatrix}a_1\\a_2\\:\\a_n\end{bmatrix} \text{ or} \begin{bmatrix}a_1 & a_2 & .. & a_n\end{bmatrix}$$

As the vectors are considered as either row matrix or column matrix, the operation of addition of vectors will have the same properties as the addition of matrices.

### 1.8.1 Linear Dependence:

The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly dependent if there exist scalars $a_1,a_2,.....a_n$ not all zero such that $a_1v_1+a_2v_2+.....+a_nv_n = 0$

### 1.8.2 Linear Independence:

The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly independent if there exist scalars $a_1,a_2,.....a_n$ such that $a_1v_1+a_2v_2+.....+a_nv_n = 0$ gives $a_1=a_2=.....=a_n = 0$ .

**Example1:** Show that the vectors u=(1,3,2), v=(1,-7,-8) and w=(2,1,-1) are linearly independent.

Proof: The vectors are said to be linearly dependent if

au+ bv +cw=0 where a, b, c are not all zero.

means a(1,3,2)+b(1,-7,-8)+c(2,1,-1)=(0,0,0) (1)

(a+b+2c, 3a-7b+c, 2a-8b-c)= (0, 0, 0)

which gives a+b+2c=0 (2)

3a-7b+c=0 (3)

2a-8b-c=0 (4)

Adding (3) and (4), we have

5a-15b=0 $\Rightarrow$ a=3b

$\therefore$ From (3) 3(3b)-7b+c=0 $\Rightarrow$ 9b-7b+c $\Rightarrow$ c=-2b

Putting a=3b and c=-2b in (2), we get

3b+b-4b=0, which is true. Giving different real value to b we get infinite non zero real values of a and c. So a, b, c are not all zero.

Hence given vectors u, v and w are linearly independent.

**Theorem 1.25:** If two vectors are linearly dependent then one of them is scalar multiple of other.

Proof: Let u, v be the two linearly dependent set of vectors. Then there exists scalars a, b(not both zero) such that

a. u + b. v = 0 (1)

Case 1. When $a \neq 0$

From (1), au = - bv $\Rightarrow$ $u = -\frac{b}{a}v$

Hence u is scalar multiple of v.

Case II. When $b \neq 0$

From (1), bv = - au $\Rightarrow$ $v=-\frac{a}{b}u$

Hence v is scalar multiple of u. Thus in both cases one of them are scalar multiple of other.

**Theorem 1.26:** Every superset of a linearly dependent set is linearly dependent.

Proof: Let $S_n = \{X_1, X_1 ...... X_n\}$ be set of n vectors which are linearly dependent.

Let $S_r = \{X_1, X_1 ......, X_n, X_{n+1} ...., X_r\}$ where r > n be any super set of $S_n$.

As $\{X_1, X_1 ...... X_n\}$ is linearly dependent set

$\therefore$ There are scalars $a_1, a_2, a_3, ......., a_n$ not all zero such that

$a_1X_1 + a_2X_2 + ....... + a_nX_n = 0$

$\Rightarrow$ $a_1X_1 + a_2X_2 + ....... + a_nX_n + 0.X_{n+1} + 0.X_{n+2} + .... + 0.X_r = 0$

As $a_1, a_2, a_3, ......., a_n$ are not all zero

$\therefore$ Set $S_r = \{X_1, X_2 ......, X_n, X_{n+1} ...., X_r\}$ is linearly dependent set.

Hence every set of linearly dependent set is linearly dependent.

**Theorem 1.27:** Every subset of linearly independent set is linearly independent.

Proof: Let $S_n = \{X_1, X_1 ...... X_n\}$ be set of n vectors which are linearly independent.

Let $S_r = \{X_1, X_1 ......, X_r\}$ where r < n be any subset of $S_n$.

As $\{X_1,X_1......X_n\}$ is linearly independent set thus

$a_1X_1+a_2X_2+.......+a_nX_n=0$ gives

$a_1=a_2=a_3,.......=a_n=0$

$a_1X_1+a_2X_2+.......+a_rX_r=0$ where $a_1=a_2=a_3,.......=a_r=0$

$\therefore$ Set $S_r=\{X_1,X_1......,X_r\}$ is linearly independent set.

Hence every subset of linearly independent set is linearly independent.

**Theorem 1.28:** If vectors $X_1,X_1......X_n$ are linearly dependent, then at least one of them may be written as linear combination of the rest.

Proof: Since the vectors $X_1,X_1......X_n$, are linearly dependent, therefore there exist scalars $a_1,a_2,a_3,.......,a_n$ not all zero, such that

$a_1X_1+a_2X_2+.......+a_nX_n=0$

Or

$a_1X_1+a_2X_2+....+a_iX_i+a_{i+1}X_{i+1}...+a_nX_n=0$

Suppose $a_i\neq 0$.

$-a_iX_i=a_1X_1+a_2X_2+....a_{i-1}X_{i-1}+a_{i+1}X_{i+1}...+a_nX_n$

or $X_i=\frac{a_1}{-a_i}X_1+\frac{a_2}{-a_i}X_2+....+\frac{a_{i-1}}{-a_i}X_{i-1}+\frac{a_{i+1}}{-a_i}X_{i+1}...+\frac{a_n}{-a_i}X_n$

Hence vector $X_i$ is a linear combination of the rest.

**Theorem 1.30:** If the set $\{X_1,X_1......X_n\}$ is linearly independent and the set $\{X_1,X_1......X_n,Y\}$ is linearly dependent, then Y is linear combination of the vectors $X_1,X_1......X_n$.

Proof: Consider the relation

$$a_1X_1+a_2X_2+.......+a_nX_n+aY=0 \qquad (1)$$

As set $\{X_1,X_1......X_n,Y\}$ is linearly dependent

$\therefore$ $a_1,a_2,a_3,.......,a_n,a$ are not all zero

We claim that $a\neq 0$. If a=0, then (1) becomes

$$a_1X_1+a_2X_2+.......+a_nX_n=0$$

As set $\{X_1,X_1......X_n\}$ is linearly independent

$\therefore$ $a_1=a_2=a_3,.......=a_n=0$

Then from (1), the set $\{X_1,X_1......X_n,Y\}$ is linearly independent which a contradiction to the given condition is. Thus a = 0 is not possible. Hence $a\neq 0$

From (1), we have $-aY=a_1X_1+a_2X_2+.......+a_nX_n$

or $Y=\frac{a_1}{-a}X_1+\frac{a_2}{-a}X_2+.......+\frac{a_n}{-a}X_n$, which proves the result.

**Theorem 1.31:** The kn-vectors $A_1,A_2,......,A_k$ are linearly dependent iff the rank of the matrix $A=[A_1,A_2,.....,A_k]$ with the given vectors as columns is less than k.

Proof: Let $x_1A_1 + x_2A_2, ...... + x_kA_k = 0$

where $x_1, x_2, ......, x_k$ are scalars

$$\Rightarrow x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_1\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + ..... + x_k\begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} = O$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 + ...... + a_{1k}x_k = 0$$

$$a_{21}x_1 + a_{22}x_2 + ...... + a_{2k}x_k = 0$$

............................................

$$a_{n1}x_1 + a_{n2}x_2 + ...... + a_{nk}x_k = 0$$

Which can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1k} \\ a_{21} & a_{22} & ... & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nk} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ AX=O

Let the vectors $A_1, A_2, ......, A_k$ be linearly dependent.

So, from the relation (i), scalars $x_1, x_2, ......, x_k$ are not all zero and thus the homogeneous system of equations given by (ii) has non-trivial solution. Hence $\rho(A)<k$ .Converse of this theorem is also true.

**Theorem 1.32:** A square matrix A is singular iff its columns (rows) are linearly dependent.

Proof: Let n be the order of the square matrix A and $A_1, A_2, ......, A_n$ be its columns.

$\therefore$ $A=[A_1, A_2, ......, A_n]$

Proceed in same way as above theorem to prove $\rho(A)<n$

Since $\rho(A)<n$, thus $|A| = 0$ and hence A is singular matrix.

Conversely, the column vectors of A are linearly dependent.

**Theorem 1.33:** The kn-vectors $A_1, A_2, ......, A_k$ are linearly independent if the rank of the matrix $A=[A_1, A_2, ....., A_k]$ is equal to k.

Proof: Proceed in the same way as above theorem to obtain AX=O. Now suppose .

Then $|A| \neq 0$ and homogeneous system of equations given by (ii) has trivial solution only.

$\therefore$ $x_1 = x_2 = ..... = x_k = 0$

Thus, the vectors $A_1, A_2, ....., A_k$ are linearly independent.

**Theorem 1.34:** The number of linearly independent solution of the equation AX=O is (n-r) where r is the rank of matrix A.

Proof: Given that rank of A is r which means A has r linearly independent columns. Let first r columns are linearly independent.

Now, $A=[C_1, C_2, .....C_r, ....., C_n]$, where $C_1, C_2, ....., C_n$ are column vectors of A.

$$\therefore [C_1, C_2, ....., C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow C_1x_1 + C_2x_2 + ..... + C_nx_n = 0 \qquad ...(i)$$

As the set $[C_1, C_2, ....., C_r]$ is linearly independent, thus each vector $C_r, C_{r+1}, ....., C_n$ can be written as linear combination of $C_1, C_2, ....., C_r$.

Now, $C_{r+1} = a_{11}C_1 + a_{12}C_2 + ..... + a_{1r}C_r$

$C_{r+2} = a_{21}C_1 + a_{22}C_2 + ..... + a_{2r}C_r$

..............................................

$C_n = a_{k1}C_1 + a_{k2}C_2 + ..... + a_{kr}C_r$ where k=n-r ...(ii)

From (i) and (ii), we get

$$X_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1r} \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, ....., X_{n-r} = \begin{bmatrix} a_{k1} \\ a_{k2} \\ \vdots \\ a_{kr} \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

Thus, AX=O has (n-r) solutions.

**Check Your Progress**

1. Find the vector p if the given vectors are linearly dependent $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ p \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Ans. p=2.

## 1.9 CHARACTERISTICS MATRIX

If A be a square matrix of order n, then we can form the matrix $[A - \lambda I]$, where I is the unit matrix of order n and $\lambda$ is scalar. The determinant corresponding to this matrix equated to zero is called the characteristic equation i.e. if $A - \lambda I$ be the matrix then

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \qquad \ldots(1)$$

is the characteristic equation of A.

On expanding the determinant (1), the characteristic equation may be written as

$(-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{n-1} \lambda + a_n = 0$

which is $n^{th}$ degree equation in $\lambda$.

The roots of (1) are called eigen values or characteristic roots or latent roots of the matrix A.

**Eigen Vectors**

We take the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}$

And if $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$ where $x_1, x_2, \ldots, x_n$ are vectors

then the linear transformation Y = AX ...(2), transforms the column vector X into the column vector Y. Generally, it is required to find such vectors which either transform it is into them selves or to a scalar multiple of themselves. If X be such a vector which is transformed into $\lambda X$ using the transformation (2) then $\lambda X = AX \Rightarrow AX - \lambda X = 0$

i.e. $\quad [A - \lambda I]X = 0 \qquad \ldots(3)$

The matrix equation (3) represents n homogeneous linear equations.

$$\begin{aligned} &(a_{11} - \lambda)x_1 + a_{12}\, x_2 + a_{13}\, x_3 + \ldots + a_{1n}\, x_n = 0 \\ &a_{21}\, x_1 + (a_{22} - \lambda)x_2 + a_{23} + x_3 + \ldots + a_{2n}\, x_n = 0 \\ &a_{31}x_1 + a_{32}\, x_2 + (a_{33} - \lambda)x_3 + \ldots + a_{3n}\, x_n = 0 \qquad \ldots(4) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{n1}\, x_1 + a_{n2}\, x_2 + (a_{n3} - \lambda)x_3 + \ldots + a_{nn} - \lambda\, x_n = 0 \end{aligned}$$

This equation (4) will have a non-trivial solution only if to co-efficient matrix is singular i.e. if the determinant $|A - \lambda I| = 0$.

This equation is also called characteristic equation of the transformation and is also the same as the characteristic equation (1) of matrix A. This characteristic equation has n roots which are eigen values of A corresponding to each root of (1), the equation (3) has non-zero solution.

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$$

which is known as an eigen vector or latent vector. So, if X is a solution of (3) then KX is also a solution, where K is an arbitrary constant. So, we see that the eigen vector corresponding to an eigen value is not unique.

**Example 1. Find the eigen values and eigen vectors of the matrices** $\mathbf{A} = \begin{bmatrix} \mathit{1} & \mathit{2} \\ \mathit{2} & \mathit{4} \end{bmatrix}$**.**

**Sol.** The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

i.e. $(1 - \lambda)(4 - \lambda) - 4 = 0 \Rightarrow \quad \lambda^2 - 5\lambda = 0 \quad \Rightarrow \quad \lambda(\lambda - 5) = 0$

i.e. $\lambda = 0, 5 \quad \therefore \quad$ eigen values of A are 0 and 5.

So, corresponding to $\lambda = 0$ eigen vectors are given by $\begin{vmatrix} 1-0 & 2 \\ 2 & 4-0 \end{vmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

i.e. $x_1 + 2x_2 = 0$ and $2x_1 + 4x_2 = 0$

i.e. single equation $x_1 + 2x_2 = 0 \quad \Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1}$ so for $\lambda = 0$ eigen vectors are $(2, -1)$ and for $\lambda =$ 5, we have $\begin{vmatrix} 1-5 & 2 \\ 2 & 4-5 \end{vmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow \quad -4x_1 + 2x_2 = 0$ and $\quad 2x_1 - x_2 = 0.$

**Properties of Eigen Values**

(I) The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal. We will prove this property for a matrix of order 3 and the method can be extended for the matrices of any finite order.

Let $\quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \ldots(1)$

Then characteristic matrix $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(\quad) + \ldots = 0 \quad \ldots(2)$

If $\lambda_1$, $\lambda_2$ and $\lambda_3$ be eigen values of A then

$|A - \lambda I| \quad = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \ldots. + (-1)^3 \lambda_1 \lambda_2 \lambda_3 \quad \ldots(3)$

Equating the co-efficients of $\lambda^2$ from (2) and (3), we get

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ which is the required result.

(II) The product of the eigen values of a matrix A is equal to its determinants. If take $\lambda = 0$ in (3) then, we get $|A - 0| = -\lambda_1\lambda_2\lambda_3$ which is the required result.

(III) If $\lambda$ is an eigen values of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of inverse matrix $A^{-1}$. If X be the eigen vector corresponding to the eigen value $\lambda$ then

$$AX = \lambda X \qquad \ldots(4)$$

Pre-multiplying (4) by $A^{-1}$, we get $A^{-1}AX = A^{-1}\lambda X$

i.e. $IX = \lambda A^{-1}X \quad \Rightarrow \quad X = \lambda(A^{-1}X) \quad \Rightarrow \quad A^{-1}X = \frac{1}{\lambda}X$

This is of the same form as that in (1) from which we get that $\frac{1}{\lambda}$ is an eigen value of the inverse matrix $A^{-1}$.

(IV) If $\lambda$ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of $A^{-1}$. As A is an orthogonal matrix so $A^{-1}$ will be same as the transpose of matrix A i.e. $A' = A^{-1}$. So, $\frac{1}{\lambda}$ is an eigen value of $A'$. But the matrix A and $A'$ have the same eigen values.

[since we know that $|A - \lambda I| = |A' - \lambda I|$ ]. Hence $\frac{1}{\lambda}$ is also an eigen value of A.

(V) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of a matrix A then $A^m$ has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ where m is a positive ineteger.

If $A_i$ be the eigen value of A and $X_i$ be the corresponding eigen vector, then

$$AX_i = \lambda_i X_i \qquad \ldots(1)$$

Consider $A^2 X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i (AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i$ similarly, we proceed and find $A^3 X_i = \lambda_i^3 X_i$ and so on such that in general we get

$$A^m X_i = \lambda_i^m X_i \qquad \ldots(2)$$

which has the same form as (1). Hence $\lambda_i^m$ is an eigen-value of $A^m$ and the corresponding eigen vector is the same as that of $X_i$.

Example 2. ***Find the characteristic roots and characteristic vectors of the matrix***

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

***Sol.*** The characteristic equation of matrix A is $| A - \lambda I | = 0$ i.e.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

i.e. $(8 - \lambda) [(7 - \lambda) (3 - \lambda) - 16] + 6[(-6) (3 - \lambda) + 8] + 2[24 - 2(7 - \lambda)] = 0$

i.e. $(8 - \lambda) [21 + \lambda^2 - 10\lambda - 16] + 6[-10 + 6\lambda] + 2[24 - 14 + 2\lambda] = 0$

i.e. $-\lambda^3 + 18\lambda^2 - 85\lambda + 40 - 60 + 36\lambda + 20 + 4\lambda = 0$

i.e. $\lambda^3 - 18\lambda^2 + 45\lambda = 0$ i.e. $\lambda = 0, 3, 15.$

$\therefore$ Corresponding to $\lambda = 0$, eigen vectors are given by

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. equations are

$$8x_1 - 6x_2 + 2x_3 = 0 \qquad \ldots(1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \qquad \ldots(2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \qquad \ldots(3)$$

From (2) and (3) we get

$$\frac{x_1}{21-16} = \frac{x_2}{-8+18} = \frac{x_3}{24-14} \quad \text{i.e.} \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

i.e. eigen vector are (1, 2, 2)

Similarly from (1) and (2) we get the same vectors

Now for $\lambda = 3$, eigen vectors are obtained from $\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

i.e.
$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. equations are

$$5x_1 - 6x_2 + 2x_3 = 0 \qquad \ldots(4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \qquad \ldots(5)$$

and
$$2x_1 - 4x_2 = 0 \qquad \ldots(6)$$

From (4) and (5), we get

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

i.e. $\dfrac{x_1}{16} = \dfrac{x_2}{8} = \dfrac{x_3}{-16} \quad \Rightarrow \quad \dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{-2}$

i.e. eigen vectors are (2, 1, –2) and for $\lambda = 15$, eigen vectors are given by

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

i.e. equation are $\quad -7x_1 - 6x_2 + 2x_3 = 0 \qquad \ldots(7)$

$6x_1 + 8x_2 + 4x_3 = 0 \qquad \ldots(8)$

and $\quad 2x_1 - 4x_2 + 2x_3 = 0 \qquad \ldots(9)$

From (7) and (8), we get

$$\frac{x_1}{12+8} = \frac{x_2}{-6-14} = \frac{x_3}{28-18} \quad \text{i.e.} \quad \frac{x_1}{20} = \frac{x_2}{-20} = \frac{x_3}{10}$$

i.e. eigen vectors are (2, –2, 1) corresponding to $\lambda = 15$.

***Example 3. Find the eigen values and eigen vectors of the matrix***

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

***Sol.*** Let the given matrix be $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

So, the characteristic equation of A is $|A - \lambda I| = 0$

i.e. $$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0 \qquad \ldots(1)$$

$\Rightarrow \quad (6 - \lambda)\,[(3 - \lambda)^2 - 1] + 2[-2(3 - \lambda) + 2] + 2[2 - 2(3 - \lambda)] = 0$

$\Rightarrow \quad (6 - \lambda)\,[9 - 6\lambda + \lambda^2 - 1] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

$\Rightarrow \quad -\lambda^3 + \lambda^2[6 + 6] - \lambda[36 - 8 + 8] + [48 - 8 - 8] = 0$

$\Rightarrow \quad \lambda^3 - 12\lambda^2 + 36\,\lambda - 32 = 0$

$\Rightarrow \quad \lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 16\,\lambda - 32 = 0$

$\Rightarrow \quad (\lambda - 2)^2\,(\lambda - 8) = 0$ i.e. $\lambda = 2, 2$ and 8.

which are the characteristic roots of (1).

Now corresponding to the eigen values $\lambda = 2, 2, 8$ the given eigen vectors are obtained from $[A - \lambda I]X = 0$.

i.e. $$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots(2)$$

(2) may be written as

$(6 - \lambda)x_1 - 2x_2 + 2x_3 = 0,$ …(A)

$-2\,x_1 + (3 - \lambda)x_2 - x_3 = 0,$ …(B)

and $2x_1 - x_2 + (3 - \lambda)x_3$ …(C)

we now, consider different cases.

***Case I. When $\lambda = 2$, then (A), (B) and (C) may be written as***

$4x_1 - 2x_2 + 2x_3 = 0$ …($A_1$)

$-2x_1 + x_2 + x_3 = 0$ …($B_1$)

$2x_1 - x_2 + x_3 = 0$ …($C_1$)

If $x_3 = 0$, then from ($A_1$) and ($B_1$), we get

$-2x_1 + x_2 = 0$ i.e. $\dfrac{x_1}{1} = \dfrac{x_2}{2}$

and so eigen vector for $\lambda = 2$, for $x_3 = 0$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

and when $x_2 = 0$, then from ($A_1$) and ($B_1$) for $\lambda = 2$,

$2x_1 + x_3 = 0 \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_3}{-2}$

$\therefore$ another eigen vector for $\lambda = 2$ is $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

***Case II. When λ = 8, equations (A), (B) and (C) become***

$-2x_1 - 2x_2 + 2x_3 = 0 \quad \ldots(A_{11})$

$-2x_1 - 5x_2 - x_3 = 0 \quad \ldots(B_{11})$

$2x_1 - x_2 - 5x_3 = 0 \quad \ldots(C_{11})$

eliminating $x_3$ from $(A_{11})$ and $(B_{11})$, we get

$x_1 + 2x_2 = 0$ i.e. $\frac{x_1}{2} = \frac{x_2}{-1}$ ...(M)

and by eliminating $x_1$ from $(A_{11})$ and $(B_{11})$, we get

$x_2 + x_3 = 0$ i.e. $\frac{x_2}{-1} = \frac{x_3}{1}$ ...(N)

Using (M) and (N), we get $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$

i.e. corresponding to $\lambda = 8$, eigen vector is $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

***Example 1. Find the eigen values and eigen vectors of the matrix***

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

***Sol.*** The characteristic equation of the given matrix is

$$|A - \lambda I| = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

i.e. $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \quad \Rightarrow \quad (\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$

i.e. eigen values are $\lambda = -3, -3, 5$

$\therefore$ If $x$, y and z be the eigen vectors. Corresponding to the eigen values $\lambda$

(I) We have $\begin{bmatrix} -1-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ 3 \end{bmatrix} = 0$ ...(1)

Now for $\lambda = 5$ we have

$-7x + 2y - 3z = 0 \qquad 2x - 4y - 6z = 0$

$-x - 2y - 5z = 0$

from (1) and (2) $\quad \frac{x}{-12-12} = \frac{y}{-6-42} = \frac{z}{28-4}$

Hence eigen vector is [1, 2, −1] $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$

(II) If $\lambda = -3$, then from (1), we get $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ which gives only one independent

$x + 2y - 3z = 0$ ...(3)

if we take $y = 0$, we get $x - 3z = 0 \Rightarrow \frac{x}{3} = \frac{y}{0} = \frac{z}{1}$

$\therefore$ for $\lambda = -3$, eigen vector is (3, 0, 1) when y = 0.

at when $z = 0$, (3) gives $x + 2y = 0 \Rightarrow \frac{x}{2} = \frac{y}{-1} = \frac{z}{0}$

i.e. eigen vector in this case is (2, −1, 0)

$\therefore$ the eigen vectors obtained are (1, 2, −1), (3, 0, 1) and (2, −1, 0)

which are the required result.

***Example 2. Find the sum and the product of eigen values of A =*** $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

***Sol.*** The characteristic equation of matrix A is $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 3 & -2 \\ -2 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

i.e. $(2-\lambda)(1-\lambda)(2-\lambda) + 3[1 + 2(2-\lambda)] + (2)(0 - \overline{1-\lambda}) = 0$

$\Rightarrow \quad (2-\lambda)(\lambda^2 - 3\lambda + 2 + 3) - 6\lambda + 15 + 2 - 2\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 11\lambda + 10 - 6\lambda + 15 + 2 - 2\lambda = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 19\lambda + 19 = 0$

$\therefore$ sum of the eigen value $\lambda_1 + \lambda_2 + \lambda_3 = -(-5) = 5$

and the product of the eigen values is $\lambda_1 \lambda_2 \lambda_3 = -19$.

**Check Your Progress**

1. Determine the charecteristics roots and the corresponding characteristics vectors of the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Ans. Characteristics roots are 0, 3, 15.

## 1.10 CAYLEY-HAMILTON THEOREM

Every square matrix satisfies its characteristic equation i.e. if A be the given square matrix of order n then its characteristic equation is $|A - \lambda I| = 0$.

i.e. $(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + K_3 \lambda^{n-3} + \ldots + K_{n-1} \lambda + K_n = 0.$ ...($M_1$)

then A will satisfy ($M_1$) i.e. $(-1)^n A^n + K_1 A^{n-1} + K_2 A^{n-2} + K_3 A^{n-3} + .. + K_n I = 0$ will hold good.

We take adjoint of the matrix $A - \lambda I$ as P i.e. $P = \text{adj.} |A - \lambda I|$. Also, each element of P is a co-factor of elements of $|A - \lambda I|$ so these co-factors are polynomials of degree $(n - 1)$ or less in $\lambda$.

So, can be split up into a number of matrices, which are co-efficients of the same power of $\lambda$ and can be written as

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + P_3 \lambda^{n-3} + \ldots + P_{n-1}\lambda + P_n \qquad \ldots(M_2)$$

where $P_1, P_2, P_3, \ldots, P_n$ are all square matrices of order n, whose elements are functions of matrix A.

By matrix property it is known that if A is a square matrix then

$A \times \text{adj. } A = |A| \times I$, where I is unit matrix of same order as that of A.

$$\therefore \qquad [A - \lambda I] \times P = |A - \lambda I| \times I$$

So using $(M_1)$ and $(M_2)$ we may write it as

$$[A - \lambda I] \times [P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + P_3 \lambda^{n-3} + \ldots + P_{n-1} \lambda + P_n]$$
$$= [(-1)^n \lambda^n + K_1 \lambda^{n-1} + \ldots + K_{n-1} \lambda + K_n]I \qquad \ldots(M_3)$$

From $(M_3)$, equating the co-efficients of powers of $\lambda$, we get

$$(-1)P_1 = (-1)^n I, \qquad \ldots(M_4)$$
$$AP_1 - P_2 = K_1 I, \qquad \ldots(M_5)$$
$$AP_2 - P_3 = K_2 I, \qquad \ldots(M_6)$$
$$AP_3 - P_4 = K_3 I, \qquad \ldots(M_7)$$
$$\ldots \qquad \ldots$$
$$AP_{n-1} - P_n = K_{n-1} I, \qquad \ldots(M_{n+3})$$

and $AP_n = K_n I. \qquad \ldots(M_{n+4})$

Next, pre-multiplying the equations $(M_4)$ by $A^n$, $(M_5)$ by $A_{n-1}$, .. and $(M_{n+3})$ by P and $(M_{n+4})$ by I, we get

$$-A^n P_1 = (-1)^n A^n, \qquad \ldots(R_1)$$
$$A^n P_1 - A^{n-1} P_2 = K_1 A^{n-1} \qquad \ldots(R_2)$$
$$A^{n-1} P_2 - A^{n-2} P_3 = K_2 A^{n-2} \qquad \ldots(R_3)$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$A^2 P_{n-1} - AP_n = K_{n-1} A \qquad \ldots(R_n)$$
$$AP_n = K_n I \qquad \ldots(R_{n+1})$$

Now, adding them, we get

$$O = (-1)^n A^n + K_1 A^{n-1} + K_2 A^{n-2} + \ldots + K_{n-1} A + K_n I \qquad \ldots(X)$$

As left hand side terms cancel.

**1.10.1 Inverse of a matrix using Cayley Hamilton theorem**

To find the inverse of any matrix A, we multiply both sides of (X) by $A^{-1}$ and get $(-1)^n A^{n-1} + K_1 A^{n-2} + K_2 A^{n-3} + \ldots + K_{n-1} + K_n A^{-1} = 0$

$$\Rightarrow \qquad A^{-1} = -\frac{1}{K_n}[(-1)^n A^{n-1} + K_1 A^{n-2} + K_2 A^{n-3} + \ldots + K_{n-1}]$$

***Example 1. Using Cayley – Hamilaton theorem, find the inverse of matrix*** $\boldsymbol{A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}}$.

***Sol.*** Let the characteristic equation be $|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(5-\lambda)(2-\lambda) - 9 = 0$ $\Rightarrow$ $10 + \lambda^2 - 7\lambda - 9 = 0$

$\Rightarrow$ $\lambda^2 - 7\lambda + 1 = 0$ ...(1)

Now $A^2 = A \times A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix}$

Consider

$$\therefore \quad A^2 - 7A + I = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} - 7\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 34-35+1 & 21-21 \\ 21-21 & 13-14+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \ldots(2)$$

$\therefore$ Cayley Hamilton is satisfied. Now, multiplying both sides of (2) by $A^{-1}$, we get

$$A - 7I + A^{-1} = 0$$

$$\therefore \quad A^{-1} = 7I - A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7-5 & -3 \\ -3 & 7-2 \end{bmatrix}$$

Hence, $$A^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

***Example 2. Using Cayley-Hamilton theorem, find the inverse of the matrix***

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

***Sol.*** The characteristic equation of the matrix A is

$$|A - \lambda I| = 0 \quad \text{i.e.} \quad \begin{bmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix} = 0$$

This may be written as $(1-\lambda)[(1-\lambda)^2 - 1] + 3(-2 + \lambda - 1] = 0$

$$-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$$

i.e. $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$ ...(1)

As (1) satisfied Cayley-Hamilton theorem, (given by question) we have

$$A^3 - 3A^2 - A + 9I = 0 \quad \ldots(2)$$

Multiplying (2) by $A^{-1}$, we get $A^2 - 3A - I + 9A^{-1} = 0$ ...(3)

So, we require

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix}$$

$$\therefore \qquad A^2 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$\therefore$ (2) may be written as $A^{-1} = \frac{1}{9}[I + 3A - A^2]$

i.e.
$$A^{-1} = \frac{1}{9}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} - \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 2/9 & -7/9 \\ 1/3 & -1/9 & -1/9 \end{bmatrix}$$

is the required inverse matrix of A.

***Example 3. Verify Cayley-Hamilton theorem for the matrix,*** $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ ***and then find*** $A^8$.

**Sol.** The characteristic equation is $|A - \lambda I| = 0$.

i.e. $\begin{bmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 1 - 4 = 0 \Rightarrow \lambda^2 = 5.$

Now, $A^2 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I$

i.e. $A^2 = 5I$.

Hence Cayley-Hamilton theorem is satisfied.

Next, $A^2 = 5I \Rightarrow A^4 = 25I \Rightarrow (A^4)^2 = (25I)^2$

$\Rightarrow \qquad A^8 = 625I.$

***Example 4. Find the characteristic equation of the matrix*** $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ ***and hence find the matrix represented by*** $A^8 - 5A^7 - 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

**Sol.** The characteristic equation of the matrix A is $|A - \lambda I| = 0$

i.e.
$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \qquad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \qquad \ldots(1)$$

Using Cayley-Hamilton theorem, we know that matrix A satisfied the eq. (1)

i.e.
$$A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(2)$$

Now, we consider the matrix

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \qquad \ldots(3)$$

and arrange it to in such a manner that (2) is used to reduce (3) in simple form.

i.e. (3) may be arranged as

$$(A^8 - 5A^7 + 7A^6 - 3A^5) + (A^4 - 5A^3 + 7A^2 - 3A) + A^2 + A + I$$
$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + 1$$

using (2)

$$= A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

which is the required result.

Since $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

***Example 5. Find the characteristic equation of the matrix*** $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. ***Show that the characteristic equation is satisfied by A and hence obtain the inverse of the given matrix.***

***Sol.*** The characteristic equation is $|A - \lambda I| = 0$.

i.e. $$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \qquad \ldots(1)$$

we have to show that A satisfies (1) i.e. $A^3 - 4A^2 - 20A - 35I = 0$ …(2)

Consider

$$A^2 = A.A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now, we consider $A^3 - 4A^2 - 20A - 35I$, which is

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60 & 232-92-140 \\ 140-60-80 & 163-88-40-35 & 208-148-60 \\ 60-40-40 & 76-36-20 & -56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Equation (2) is satisfied and $A^{-1} = \dfrac{1}{35}[A^2 - 4A - 20I]$

i.e. $$A^{-1} = \frac{1}{35}\left\{\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right\}$$

$$= \frac{1}{35}\begin{bmatrix} 20-4-20 & 23-12 & 23-28 \\ 15-16 & 23-8-20 & 37-12 \\ 10-14 & 9-8 & 14-4-20 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

i.e. $A^{-1} = \dfrac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$ is the required result..

## 1.12 SUMMARY

In this unit we have learned about

- A matrix is said to be symmetric if $A^T = A$.
- A necessary and sufficient condition for a matrix to be skew symmetric is that $A^T = -A$.
- Diagonal elements of a skew symmetric matrix are zero.
- A matrix is said to be Hermitian if $A^\theta = A$.
- A necessary and sufficient condition for a matrix to be skew Hermitian is that $A^\theta = -A$.

- Diagonal element for a skew Hermitian matrix are either zero or purely imaginary.
- The rank of a matrix is the largest order of any non zero minor of the matrix.
- The elementary operation is operation on any row/column like interchange of row/column, multiplication of all element of row/column by a number, addition of any row or column. The rank of a matrix is not changed when we apply elementary operation on a matrix.
- Two matrices are said to be equivalent if one can be obtained from other by applying finite number of elementary operation. Equivalent matrices have same rank.
- A matrix is said to be row echelon matrix if the leading entry of each non zero row is unity, the number of zero before the leading entry is less than the number of zero in the succeeding rows and the non zero rows precede the zero rows. A row echelon matrix is said to be row reduced echelon matrix if each column containing the leading entry of a row has all the other elements as zero.
- A matrix of order $m \times n$ and rank r is equivalent to the matrix $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ in the normal form.
- The matrix obtained by applying a single elementary operation on identity matrix is called elementary matrix.
- The rank of the product of two matrices cannot exceed the rank of either matrix. The rank, row rank and column rank are all equal.
- If A is $n \times n$ matrix, then the matrix $A-\lambda I$, for some scalar $\lambda$ is called characteristics matrix of A.
- The determinant of the matrix $A-\lambda I$ is a non null polynomial of degree n in $\lambda$ and is called characteristics polynomial of matrix A.
- The equation $\lfloor A-\lambda I \rfloor = 0$, for some scalar $\lambda$ is called the characteristics equation of matrix A and its roots , $\lambda_1, \lambda_2, ...., \lambda_n$ are called the characteristics roots of matrix A.
- If $\lambda$ is the characteristic root of an $n \times n$ matrix A, then any solution of the equation $AX=\lambda X$ except $X=0$ is called a characteristic vector of matrix A.
- If A is non singular, then the eigen value of $A^{-1}$ is the reciprocal of the eigen value of A.
- The eigen value of diagonal matrix is the diagonal elements of the matrix.
- The eigen value of triangular matrix is the diagonal elements of the matrix.
- If a is eigen value of a non singular matrix A, then $\frac{\lfloor A \rfloor}{a}$ is an eigen value of adj. A.
- The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.
- The distinct roots of the characteristic equation $\phi(\lambda)=0$ of a matrix A are also the distinct roots of the minimal equation $m(\lambda)=0$ of the matrix A.

## 1.13 KEY TERMS

- **Symmetric matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.
- **Skew Symmetric matrix:** If a square matrix A has its elements such that $a_{ij} = -a_{ji}$ for i and j and the leading diagonal elements are zeros, then matrix A is known as skew matrix.

- **Hermitian Matrix:** A square matrix $A=[a_{ij}]$ over the complex numbers is said to be Hermitian if the transposed conjugate of the matrix is equal to the matrix itself i.e. $A^{\theta}=A$ .
- **Skew Hermitian Matrix:** A square matrix $A=[a_{ij}]$ over the complex numbers is said to be Skew Hermitian if the transposed conjugate of the matrix is equal to the negative of matrix itself i.e. $A^{\theta}=-A$ .
- **Rank of a Matrix:** A non zero matrix has rank r if every minor of order r+1 vanishes and it has at least one non non zero minor off order r.
- **Linear Dependence and Independence of row and column Matrices:** The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly dependent if there exist scalars $a_1,a_2,.....a_n$ not all zero such that $a_1v_1+a_2v_2+.....+a_nv_n=0$ .The set of vectors $\{v_1,v_2,.....v_n\}$ are said to be linearly independent if there exist scalars $a_1,a_2,.....a_n$ such that $a_1v_1+a_2v_2+.....+a_nv_n=0$ gives $a_1=a_2=.....=a_n=0$ .

## 1.14 QUESTION AND EXERCISE

1. Define inverse of a square matrix and show that inverse of a matrix is unique if it exists
2. Prove that square matrix A is invertible iff A is non singular.
3. Find the adjoint of the matrix $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and verify A(adj. A)=(adj.A)A= $|A|I_3$ .
4. Calculate the inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ if exists.
5. If A and B are square matrix of same order and a is non singular, then prove that $\lfloor A^{-1}BA \rfloor = \lfloor B \rfloor$.
6. Solve the system of equation using matrix method

(i) $\begin{array}{l} 2x-3y+z=9 \\ x+y+z=6 \\ x-y+z=2 \end{array}$ (ii) $\begin{array}{l} x+z=7 \\ 2x+y=7 \\ 3x+2y+z=17 \end{array}$

7. If A is any square matrix, prove that $A^TA$ and $AA^T$ are both symmetric.
8. Prove that $B^TAB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.
9. If A is skew symmetric matrix of order n, then show that adj. A is symmetric or skew symmetric according as n is odd and even.
10. Express $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrix.
11. Show all positive odd integral power of a skew symmetric matrix are skew symmetric while positive even integral power are symmetric.
12. If A and B are Hermitian, show that AB +BA is Hermitian and AB is Hermitian iff AB = BA.

13. Find the rank of matrix $\begin{bmatrix} 1 & -3 & 4 & 6 \\ 9 & 1 & 2 & 0 \end{bmatrix}$.

14. Find the rank of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$, a, b, c being real.

15. If A is a square matrix of rank n -1, show that adj. $A \neq O$.

16. Reduce the following matrix to normal form $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 2 \end{bmatrix}$.

17. Reduce the matrix $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ to $\left[I_3 \; O\right]$. Hence find $\rho(A)$.

18. Express $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as product of elementary matrices.

19. For the given matrix A, find non singular matrix P and Q such that PAQ is in normal form and hence determine the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

20. Prove that the set of vector (0, 2, -4), (1,-2,-1), (1,-4, 3) is linearly dependent.

21. Find p if the vectors $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly dependent.

22. Verify Cayley Hamilton theorem for the given matrix and also find $A^{-1}$ also:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}.$$

## 1.15 FURTHER READING

L.N. Herstein *Topic in Algebra, ,* Wiley Eastern Ltd. New Delhi, 1975

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**UNIT – II**

# SYSTEM OF LINEAR EQUATIONS



## 2.0 INTRODUCTION

In this chapter we will use these concepts to study the solutions of systems of linear equations. Here, we will concentrate on the solution of system of homogeneous as well as non-homogeneous linear equations. We also learn orthogonal and unitary matrices.

## 2.1 OBJECTIVES

After going through this unit you will be able to:

- Determine whether the system of non homogeneous and homogeneous linear equation is consistent or inconsistent..
- Solve non homogeneous and homogeneous system of linear equations.

## 2.2 LINEAR SYSTEM OF EQUATIONS

### 2.2.1 System of Non Homogeneous Linear Equation

If

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots(1)$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_n$$

be given system of m linear equations then (1) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}$$

$$\Rightarrow \quad AX = B \qquad \text{and} \qquad C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} : b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} : b_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} : b_m \end{bmatrix}$$

then [A : B] or C is called augmented matrix. Sometime we also write A : B for [A : B]

***Consistent Equations.***

(i) If rank of A = rank of [A : B] and there is unique solution when rank of A=rank of [A : B]=n

(ii) rank of A = rank of [A : B] = r < n.

***Inconsistent Equations.***

If rank of A ≠ rank of [A : B] i.e. have no solution.

***Example 1. Discuss the consistency of the following system of equation $2x + 3y + 4z = 11$, $x + 5y + 7z = 15$, $3x + 11y + 13z = 25$, if consistent, solve.***

***Sol.*** The augmented matrix $[A : B] = \begin{bmatrix} 2 & 3 & 4:11 \\ 1 & 5 & 7:15 \\ 3 & 11 & 13:25 \end{bmatrix}$

$R_{12}$ operation is done so $\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 2 & 3 & 4:11 \\ 3 & 11 & 13:25 \end{bmatrix}$

Next operating $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & -7 & -10:-19 \\ 0 & -4 & -8:-20 \end{bmatrix}$$

Again, operating $R_2 \to -\frac{1}{7} R_2$ and $R_3 \to -\frac{1}{4} R_3$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & 1 & \frac{10}{7}:\frac{19}{7} \\ 0 & 1 & 2:5 \end{bmatrix}$$

Next operating $R_3 \to R_3 - R_2$, we get

$$\sim \begin{bmatrix} 1 & 5 & 7:15 \\ 0 & 1 & \frac{10}{7}:\frac{19}{7} \\ 0 & 0 & \frac{4}{7}:\frac{16}{7} \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + 5y + 7z &= 15 \\ y + \frac{10}{7}z &= \frac{19}{7} \\ \frac{4}{7}z &= \frac{16}{7} \end{aligned} \qquad \ldots(M)$$

From which we get rank of A = 3 as well as rank of A : B = 3. Hence the system of equations is consistent and has unique solution $\frac{4}{7}z = \frac{16}{7} \Rightarrow z = 4$

And $y + \frac{10}{7}z = \frac{19}{7} \Rightarrow \quad y + \frac{10}{7} \times 4 = \frac{19}{7} \quad \Rightarrow y = -\frac{21}{7} = -3$

And from (M), we have $x + 5y + 7z = 15 \Rightarrow x = 2$

i.e. we have the solution $x = 2$, $y = -3$ and $z = 4$, which is the required result.

***Example 2. Test the following equations for consistency and hence solve these equations $2x - 3y + 7z = 5$, $3x + y - 3z = 13$ and $2x + 19y - 47z = 32$.***

***Sol.*** The above equations may be written as AX = B.

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

Operating $R_2 \to 2R_2 - 3R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

Next, we operate $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & +22 & -54 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 27 \end{bmatrix}$$

This indicate the rank of A = 2 which is less than 3 (the number of variables) i.e.

$\rho(A) = 2 < 3$

So, the given equations are not consistent and so infinite number of solutions can be obtained.

***Example 3. Show that if $\lambda \neq -5$, the system of equation $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solution, in each case.***

***Sol.*** The given equations are

$$\left.\begin{aligned} &3x - y + 4z = 3, \\ &x + 2y - 3z = -2 \\ \text{and}\quad &6x + 5y + \lambda z = -3 \end{aligned}\right\} \quad \ldots(1)$$

If $A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$ such that AX = B from (1)

Then augmented matrix $A : B = \begin{bmatrix} 3 & -1 & 4:3 \\ 1 & 2 & -3:-2 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$

Operating $R_{12}$(i.e. interchanging $R_1$ and $R_2$)

$$A : B = \begin{bmatrix} 1 & 2 & -3:-2 \\ 3 & -1 & 4:3 \\ 6 & 5 & \lambda:-3 \end{bmatrix}$$

Now operating $R_2 - 3R_1$ [i.e. $R_2$, 1(–3)] and $R_3$, 1(–6) i.e. $R_3 - 6R_1$, we get

$$A : B \sim \begin{bmatrix} 1 & 2 & 3 & : -2 \\ 0 & -7 & 13 & : 9 \\ 0 & -7 & \lambda+18 & : 9 \end{bmatrix}$$

Next, $R_3 - R_2$[(i.e. $R_3$, 2(−1)], we get

$$\sim \begin{bmatrix} 1 & 2 & -3 & : -2 \\ 0 & -7 & 13 & : 9 \\ 0 & 0 & \lambda+5 & : 0 \end{bmatrix} \quad \ldots(2)$$

If $\lambda = -5$, then rank of A becomes $\rho(A) = 2$ which is less than 3, (the number of unknowns) and hence the equations will be consistent and will have infinite number of solutions

Next, operating, $R_1 + \frac{2}{7}R$, we get

$$\sim \begin{bmatrix} 1 & 0 & 5 : \frac{4}{7} \\ 0 & -7 & 13 : 9 \\ 0 & 0 & \lambda+5 : 0 \end{bmatrix} \text{ from this matrix, if } \lambda \neq -5$$

then rank is 3 and the equation will be consistent and we get

$$x + \frac{5}{7}z = \frac{4}{7} \; ; \; -7y + 13z = 9 \text{ and } (\lambda + 5)z = 0 \text{ i.e. } z = 0$$

$$\Rightarrow \quad -7y = 9 \Rightarrow \quad y = -\frac{9}{7} \text{ and } x + 0 = \frac{4}{7} \text{ i.e. } x = \frac{4}{7}.$$

i.e. unique solution is $x = \frac{4}{7}$, $y = -\frac{9}{7}$, $z = 0$, which is required result.

If $\lambda = -5$, then from (2), we have $x + 2y - 3z = -2$, $-7y + 13z = 9$ …(3)

If we take z = k than from (3),

$$y = \frac{13k - 9}{7} \quad \text{and} \quad z = \frac{3k + 2\left(\frac{13k-9}{7}\right) - 2}{3} - \frac{4 - 5k}{7}$$

***Example 4. Examine whether the following equations are consistent and solve them if they are consistent $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$ and $6y - 18z + 1 = 0$.***

***Sol.*** The above equations may be written in the form

$$AX = B \text{ which is } \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} \quad \ldots(1)$$

Now the augmented matrix may be written as

$$A : B = \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \quad \ldots(2)$$

Operating $R_2 \to R_2 - 3R_1$, we get

$$A : B \sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

Now, operating $R_3 \rightarrow R_3 - 3R_2$, we get

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

Hence rank of A = $\rho(A) = 2$ and $\rho(A : B) = 3$. So, $\rho(A) = 2 < 3$ (number of variables). This indicated that given equation are in consistent and so it has no unique solution.

***Example 5. Solve the following system of equations by matrix method $x + y + z = 8$, $x - y + 2z = 6$ and $3x + 5y - 7z = 14$.***

***Sol.*** The above equations written in the form AX = B.

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 6 \\ 14 \end{bmatrix}$$

So, we may write augmented matrix as

$$A : B = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 2 & : & 6 \\ 3 & 5 & -7 & : & 14 \end{bmatrix} \quad \ldots(1)$$

Operating $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we have

$$A : B \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 2 & 10 & : & 10 \end{bmatrix} \quad \ldots(2)$$

Again $R_3 \rightarrow R_3 + R_2$, we have

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 0 & -9 & : & -12 \end{bmatrix}$$

this implies that

$$x + y + z = 8$$
$$-2y + z = -2 \quad \ldots(3)$$

and $$-9z = -12$$

$$\Rightarrow \quad z = \frac{4}{3} \text{ and } 2y = z + 2 = \frac{4}{3} + 2 = \frac{10}{3} \quad \therefore \quad y = \frac{5}{3}$$

Using 1st equation of (3), we get $x + y + z = 8$

$$\Rightarrow \quad x + \frac{5}{3} + \frac{4}{3} = 8 \quad \Rightarrow \quad x = 8 - 3 = 5$$

From (2) we see that $\rho(A) = 3$ = number of variables so, the system of equations are consistent and solutions are $x = 5$, $y = \frac{5}{3}$, $z = \frac{4}{3}$.

***Example 6. Determine for what values of $\lambda$ and $\mu$ the following equations have (i) no solution ii) a unique solution (iii) infinite number of solution : $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$.***

***Sol.*** The above equations may be written in the form AX = B.

i.e.
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

The augmented matrix [A : B] =
$$\begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get
$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

Again operating $R_3 - R_2$, we get
$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

$\Rightarrow$we get $x + y + z = 6$, $y + 2z = 4$ and $(\lambda - 3)z = \mu - 10$.

(i) If $R(A) \neq R[A : B]$ i.e. if $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, then rank of A $\neq$ rank of [A : B]. Since $\rho(A) = 2$ and $\rho(A : B) = 3$. The equation have no solution.

(ii) The equations have unique solution if rank of A = rank of [A : B] = 3, i.e. if $\lambda - 3 \neq 0$ and $\mu - 3 \neq 0$.

(iii) If $\rho(A) = \rho(A : B) = 2$ i.e. when $\lambda - 3 = 0$ and $\mu - 10 = 0$ i.e. when $\lambda = 3$ and $\mu = 10$. Then these are infinite number of solution.

**2.2.2 System of Homogeneous Linear Equations**

If

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \quad \ldots(1)$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n =$$

be given system of m linear equations then (1) may be written as AX=O

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}$$

Here A is called the coefficient matrix and the given system of equations AX=O is called linear homogeneous system of equations.

**Working rule for determining solution of m homogeneous equations in n variables.**

Firstly we find the rank of coefficient matrix A. Then

1. There is only a trivial solution which is $x_1=x_2=.....=x_n=0$ if $\rho(A) = n$ .
2. A can be reduced to a matrix which has (n-r) zero rows and r non zero rows and if $\rho(A) < n$ so the system is consistent and has infinite number of solutions.

Thus, the given system of equations has a non- trivial solution iff $|A| = 0$

**Example 1:** Solve the following system of equations

$$x - y + z = 0$$
$$x + 2y - z = 0$$
$$2x + y + 3z = 0$$

Solution. Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

Operating $R_2 \to R_2+(-R_1)$ and $R_3 \to R_3+(-2)R_1,$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3+(-R_2)$, $A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$

Operating $R_2 \to R_2 \times \left(\frac{1}{3}\right)$ and $R_3 \to R_3 \times \left(\frac{1}{3}\right)$

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\rho(A)=3$ = number of variables and hence the given system of equations has only trivial solution, x = y = z = 0.

**Example:** Solve the following system of equations:

$$x - y + 2z - 3w = 0$$
$$3x + 2y - 4z + w = 0$$
$$4x - 2y + 9w = 0$$

Solution: Writing the given equations in the matrix form, we have

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or AX=O, where $A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & 2 & -4 & 1 \\ 4 & -2 & 0 & 9 \end{bmatrix}$

Operating $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -10 & 10 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_2 \to R_2\left(\frac{1}{5}\right)$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 2 & -8 & 21 \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$,

$$A \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -4 & 17 \end{bmatrix}$$

$\therefore$ $\rho(A)=3$ , Here n = 4 (the number of unknowns)

Now $\rho(A)<4$ . Thus the system of equations has infinite solutions. The solutions will contain 4 – 3=1 arbitrary constant.

Equation corresponding to the matrix are

$$x - y + 2z - 3w = 0 \qquad (1)$$

$$y - 2z + 2w = 0 \qquad (2)$$

$$-4z + 17w = 0 \qquad (3)$$

From (3), $z = \frac{17}{4}w$

$\therefore$ From (2), $y - \frac{17}{2}w + 2w = 0 \Rightarrow y = \frac{13}{2}w$

$\therefore$ From (1), $x - \frac{13}{2}w + \frac{17}{2}w - 3w = 0 \Rightarrow x = w$

Putting w = k, we get $x = k$, $y = \frac{13}{2}k$, $z = \frac{17}{4}k$ , which is the general solution, where k is an arbitrary parameter.

**Check Your Progress**

1. Solve the following system of liear equation

$x-y+z=0$

$x+2y-z=0$

$2x+y+3z=0$

Ans. x = y = z = 0.

2. Find the values of a and b for which the following system of linear equations

$2x+by-z=3$

$5x+7y+z=7$.

$ax+y+3z=a$

Ans. a = 1 and b = 3.

## 2.3. **Orthogonal and Unitary matrices**

### 2.3.1 Orthogonal Matrix

Any square matrix A is said to be orthogonal if $AA'=A'A=I$, this indicates that the row vectors (column vectors) of an orthogonal matrix A are mutually orthogonal unit vectors.

### 2.3.2 Unitary Matrix

Any square A with complex elements is said to be unitary if $A.A^{\theta}=A^{\theta}A=I$.

## 2.4 SUMMARY

- The matrix A and B together form a matrix $[A:B]$ termed as augmented matrix which is denoted as

$$[A:B]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n}:b_1 \\ a_{21} & a_{22} & \dots & a_{2n}:b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn}:b_m \end{bmatrix}$$

- The system of linear equation $AX=B$ is consistent if the coefficient matrix A and the augmented matrix $[A:B]$ have the same rank i.e. $\rho(A)=\rho[A:B]$.
- If A is a non singular square matrix of order n and X,B are matrices of order $(n\times 1)$, then the system $AX=B$ possesses a unique solution.

## 2.5 KEY TERMS

- **System of Non Homogeneous Linear Equation:**

**Consistent Equations.**

(i) If rank of A = rank of [A : B] and there is
unique solution when rank of A = rank of [A : B] = n

(i) rank of A = rank of [A : B] = r < n.

**Inconsistent Equations.**

If rank of A ≠ rank of [A : B] i.e. have no solution.

- **System of Homogeneous Linear Equations:**

Firstly we find the rank of coefficient matrix A. Then

1. There is only a trivial solution which is $x_1=x_2=.....=x_n=0$ if $\rho(A)=n$.

2. A can be reduced to a matrix which has (n-r) zero rows and r non zero rows and if $\rho(A) < n$ so the system is consistent and has infinite number of solutions.

Thus, the given system of equations has a non- trivial solution iff $|A| = 0$.

## 2.6 QUESTION AND EXERCISE

**1.** Solve the system of linear equation:

$\lambda$x+2y-2z=1

4x+2$\lambda$y-z=2 , considering special the case when $\lambda = 2$.

6x+6y+$\lambda$z=3

2. Show that the only real value of $\lambda$ for which the equations

x+2y+3z=$\lambda$x

3x+y+2z=$\lambda y$ , have a non zero solution is 6.

2x+3y+z=$\lambda z$

## 2.7 FURTHER READING

L.N. Herstein *Topic in Algebra,* Wiley Eastern Ltd. New Delhi, 1975

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UNIT – III

# EQUATION AND POLYNOMIAL

**STRUCTURE**



## 3.0 INTRODUCTION

In this unit, we will learn synthetic division . We will also find the relation between the roots and the co-efficient of an equation. In this unit we will transform equations in another equations whose roots are related. Transformation of an equation into another is a very useful as we can connect the roots of the new equation with that of the given equation or convert the co-efficient of the new equation in particular forms. Also the transformed equation may be easier to solve and having solved the transformed equation we can find out the roots of the given equation with the help of the relation between the roots of the given equation and the transformed equation.

## 3.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Know about polynomials and their co-efficient, terms and value.
- Know about degree and roots of complete and incomplete equations.
- Learn synthetic division.
- Find the relation between the roots and co-efficient of an equation.
- Find a condition so that the roots of a given equation satisfy a given relation and the common roots of two equations.
- Find multiple roots of an equation and the common roots of two equations.
- Transform an equation into another having roots with sigh.
- Changed, roots multiplied by a number, roots as the reciprocal of the roots of the given equation.
- Remove the fractional co-efficient removed of the given equation.
- Diminish the roots of an equation by a given number.
- Remove particular terms of the given equation .
- To find an equation having roots as squared differences of the roots of the equation $x^3$.
- To find an equation having roots as squared differences of the roots of the equation.

## 3.2 POLYNOMIAL

An expression in x of the form $a_0x^n + a_1x^{n-1} + ... + a_n$ where $a_0, a_1, ..., a_n$ are constants known as the coefficients and x is a variable. We shall denote a polynomial by $f(x)$ or $g(x)$ etc**.** A polynomial $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ is said to be of degree n when $a_0 \neq 0$.

- If all the coefficients $a_0, a_1, ..., a_n$ in $f(x)$ are real then polynomial is said to be **real polynomial**.
- If all the coefficients $a_0, a_1, ..., a_n$ in $f(x)$ are zero then polynomial is said to be **zero polynomial**.
- If we substitute x=a in $f(x)$ then the number $f(a)$ is called **value of the polynomial** $f(x)$ for x=a.
- The number x=a is called **zero of the polynomial** $f(x)$ if and only if $f(a) = 0$.
- The highest index of variable x occurring in the terms of polynomial $f(x)$ is called **degree of** $f(x)$.

**Example**: The polynomial $f(x) = 5 + 7x^2 + 3x^3$ is of degree three.

### 3.2.1 Division Algorithm

If $f(x)$ and $g(x)$ are two nonzero polynomial, then there exist unique polynomial $q(x)$ and that $f(x) = q(x).g(x) + r(x)$, where $r(x)$ is either a zero polynomial or degree of $r(x)$< degree of $g(x)$. Where $q(x)$ is called quotient and $r(x)$ is called remainder when $f(x)$ is divided by $g(x)$.

- Degree of $r(x)$ = degree of $f(x)$ - degree of $g(x)$.
- If degree of $g(x)$ =1 then, either $r(x)$ =0 or degree of $r(x)$ is zero means a constants polynomial.

### 2.2.2 Remainder Theorem

**Theorem 2.1:** If a polynomial $f(x)$ is divided by x-a, then the remainder is equal to $f(a)$.

**Proof:** Let $q(x)$ be quotient and R be the remainder then using division algorithm for polynomial $f(x)$ and x-a, we have

$f(x) = q(x)(x-a) + R$

By putting x=a, we get

$f(a) = q(a)(a-a) + R \Rightarrow R = f(a)$ =Remainder

**Example 1:** Find the remainder when polynomial $2x^4 - x^3 - 6x^2 + 4x - 8$ is divided by x+2.

**Sol**: $f(x) = 2x^4 - x^3 - 6x^2 + 4x - 8$

Comparing x+2 with x-h, we get h=-2

$\therefore$ Now Remainder R= $f(h) = f(-2)$ [ $\because$ $h = -2$ ]

= $2(-2)^4 - (-2)^3 - 6(-2)^2 + 4(-2) - 8$

=32+8-24-8-8=0

Thus remainder is zero.

Note: When Remainder R=0, we say that $f(x)$ is exactly divisible by x+2.

### 2.2.3 Factor Theorem

**Theorem 2.2:** If h is a root of the equation f(x)=0, then (x-h) is a factor of f(x) and conversely.

Proof: Using Division algorithm for f(x) and (x-h), we have

f (x)=q(x)(x-h)+R (1)

where q(x) is quotient and R is remainder.

Now as h is a root of equation f(x) =0 $\Rightarrow$ f(h)=0 (2)

from (1), we have

f(h)= q(h)(h-h)+R

f(h)=R, but from (2)

R=0, using in (1), we have

f(x)=q(x)(x-h) , this imply (x-h) is factor of f(x).

Converse: Do yourself.

## 3.3 GENERAL EQUATION

If $f(x)$ is a polynomial then $f(x)$=0 is called **general equation.**

### 3.3.1 Degree of an Equation:

The degree of an equation $f(x) = 0$ is highest index of variable x occurring in the terms of polynomial equation $f(x) = 0$. The polynomial equation of 1st degree, 2nd degree, 3rd degree, 4th degree, are respectively known as Linear, Quadratic, Cubic, biquadratic Equation.

### 3.3.2 Complete and Incomplete Equations

A General Equation of degree n is said to be complete if it contains all power of the variable x from 0 to n. For example $a_0x^3 + a_1x^2 - a_3x + a_4 = 0$ is **complete equation** of third degree.

If any of the powers of the variable are missing from an equation of degree n is called **incomplete equations**. For example $a_0x^3 + a_1x^2 + a_4 = 0$ is **incomplete equation** of third degree. [ $\because$ The Term $-a_3x$ is missing.]

### 3.3.3 Root of an Equation

The value of x for which $f(x)$ vanishes is called root of equation $f(x)=0$ .For example if $f(h)=0$ , then h is called root of the equation $f(x)=0$ .

## 3.4 SYNTHETIC DIVISION

It is a rule of coefficient detached with the help of which we can find quickly the quotient and the remainder when a polynomial f(x) is divided by a polynomial of the form x-a or ax+b ( $a \neq 0$ )

The rule is explained with help of a example:

Let $3x^4-5x^3+10x^2+11x-61$ is divided by $x-3$

(a) In the first line write down in descending order the co-efficient 3, -5, 10, 11 and -61 of given equation (coefficient of missing terms are to be written as zeros).

(b) Put the divisor (x-3) =0 and find the value of x i.e., x=3, which is called **multiplier** and may be written at the left hand corner separated by a vertical line .

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 3 | -5 | 10 | 11 | -61 | |
| | | 9 | 12 | 66 | 231 | |
| | | 3 | 4 | 22 | 77 | 170=Remainder |

Hence the remainder=170 and the quotient is $3x^3+4x^2+22x+77$ .

### 3.4.1 To find the Quotient and Remainder, when $f(x)$ is divided by $(ax-b)$.

The divisor can be written as $a\left(x-\frac{b}{a}\right)$

Divide $f(x)$ by $\left(x-\frac{b}{a}\right)$, let $q(x)$ be the quotient and R the remainder.

$$\therefore \qquad f(x)=q(x).\left(x-\frac{b}{a}\right)+R=q(x).\frac{(ax-b)}{a}+R=\frac{1}{a}.q(x)(ax-b)+R$$

This shows that when $f(x)$ is divided by $(ax-b)$ instead of $\left(x-\frac{b}{a}\right)$, the remainder R remain unchanged, whereas the quotient $q(x)$ is to be divided by "a " the coefficient of x in the divisor $(ax-b)$.

**Example 2:** Find the quotient and remainder, when $2x^4-5x^2+11x-7$ is divided by $(2x-3)$.

Sol. The divisor $(2x-3)$ can be written as $2\left(x-\frac{3}{2}\right)$

We first divide by $\left(x-\frac{3}{2}\right)$.

Put $x-\frac{3}{2}=0 \quad \therefore \quad x=\frac{3}{2}$ is a multiplier.

Write down the co-efficient of the given polynomial in descending order and write down zero coefficient in place of missing terms i.e.,

| | | | | | |
|---|---|---|---|---|---|
| $\frac{3}{2}$ | 2 | 0 | -5 | 11 | -7 |
| | | 3 | $\frac{9}{2}$ | $-\frac{3}{4}$ | $\frac{123}{8}$ |
| | 2 | 3 | $-\frac{1}{2}$ | $\frac{41}{4}$ | $\frac{67}{8}$ |

This gives the remainder and quotient when $f(x)$ is divided by $\left(x-\frac{3}{2}\right)$.

Therefore, the remainder and quotient when $f(x)$ is divided by $(2x-3)$.

Required Remainder= $\frac{67}{8}$ [Remainder remain same]

Required quotient= $\frac{1}{2}q(x)$

$$= \frac{1}{2}(2x^3+3x^2-\frac{1}{2}x+\frac{41}{4}) = x^3+\frac{3}{2}x^2-\frac{1}{4}x+\frac{41}{8}.$$

**Problems to Check Your Progress**

**1.** Using synthetic division find out $f(2)$ , where $f(x)=x^4-12x^2+40x-71$.

2. Given that -4 is a root of the equation $2x^3+6x^2+7x+60=0$, find the other roots.

## 3.5 RELATION BETWEEN THE ROOT AND COEFFICIENT OF AN EQUATION.

Let the given equation be

$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_{n-1}x+a_n=0$

Let $\alpha_1,\alpha_2,\alpha_3,....,\alpha_n$ be its n roots.

$\therefore \quad f(x)=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$.

Now equating the two expressions for the polynomial $f(x)$, we have

$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_{n-1}x+a_n=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$.

$= a_0\left[x^n-\sum\alpha_1x^{n-1}+\sum\alpha_1\alpha_2x^{n-1}-\sum\alpha_1\alpha_1\alpha_2x^{n-1}+.....+(-1)^n\alpha_1\alpha_1\alpha_2..\alpha_n\right]\ldots$ (1)

where, $\sum\alpha_1$ = the sum of all the roots.

$\sum\alpha_1\alpha_2$ = the sum of product of the roots taken two at a time.

$\sum\alpha_1\alpha_2\alpha_3$ = the sum of the products of the roots taken three at a time.

Equating the co-efficient of like powers of x on both sides of (1), we get

$$-a_0\sum\alpha_1=a_1 \Rightarrow \sum\alpha_1=-\frac{a_1}{a_0};$$

$$-a_0\sum\alpha_1\alpha_2=a_2 \Rightarrow \sum\alpha_1\alpha_2=(-1)^2\frac{a_2}{a_0};$$

$$-a_0\sum \alpha_1\alpha_2\alpha_3 = a_3 \Rightarrow \sum \alpha_1\alpha_2\alpha_3 = (-1)^3\frac{a_3}{a_0};$$

$$a_0(-1)^n\alpha_1\alpha_2\alpha_3......\alpha_n = a_n \Rightarrow \alpha_1\alpha_2\alpha_3......\alpha_n = (-1)^n\frac{a_n}{a_0};$$

Thus, above results gives the required relations between the roots and coefficients of the equation $f(x)=0$.

Sum of the products of roots taken r at a time is $\sum \alpha_1\alpha_2\alpha_3...\alpha_r = (-1)^r\dfrac{\text{coeff. of } T_{r+1}}{\text{coeff. of } T_1}$

Particular Case:

(i) If α, β are roots of the equation

$ax^2+bx+c=0$, then $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

(ii) If $\alpha,\beta$ and $\gamma$ are roots of the equation

$ax^3+bx^2+cx+d=0$, then $\alpha+\beta+\gamma=-\dfrac{b}{a}$

$\alpha\beta+\beta\gamma+\gamma\alpha=(-1)^2\dfrac{c}{a}$, $\alpha\beta\gamma=(-1)^3\dfrac{d}{a}$

(iii) If $\alpha,\beta,\gamma$ and $\delta$ are roots of the equation

$ax^4+bx^3+cx^2+dx+e=0$, then $\alpha+\beta+\gamma+\delta=-\dfrac{b}{a}$

$\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha=(-1)^2\dfrac{c}{a}$, $\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha=(-1)^3\dfrac{d}{a}$ and $\alpha\beta\gamma\delta=(-1)^4\dfrac{e}{a}$.

### 3.5.1 Solution of Polynomial Equation having Condition on the Roots

We can solve given polynomial equation with the help of relations $\sum\alpha$, $\sum\alpha_1\alpha_2$ ….., where $\alpha_i$ denotes the roots of given equation.

In case of cubic equation:

(i) If the roots are given in A.P. can be taken as $\alpha-\beta,\alpha,\alpha+\beta$.

(ii) If the roots are given in G.P. can be taken as $\dfrac{\alpha}{\beta},\alpha,\alpha\beta$.

In case of bi-quadratic equation:

(i) If the roots are given in A.P. can be taken as $\alpha-3\beta,\alpha-\beta,\alpha+\beta,\alpha+3\beta$.

(ii) If the roots are given in G.P. can be taken as $\dfrac{\alpha}{\beta^3},\dfrac{\alpha}{\beta},\alpha\beta,\alpha\beta^3$.

**Example 3:** Solve the equation $4x^3+16x^2-9x-36=0$, the sum of two of the roots being zero.

**Solution:** The given equation is $4x^3+16x^2-9x-36=0$

Let the roots be $\alpha,\beta,\gamma$ such that $\alpha+\beta=0$

Now $\quad\quad \alpha+\beta+\gamma=-\dfrac{16}{4}=-4$

Since $\alpha+\beta=0 \quad \Rightarrow \quad \gamma=-4$

Thus one root of equation is -4 and hence x + 4 is a factor of given equation

Dividing the given equation by x + 4, we have

Depressed equation is $4x^2-9=0 \quad \Rightarrow \quad x^2=\frac{9}{4} \quad \Rightarrow x=\pm\frac{3}{2}$

Hence the roots are -4, $\pm\frac{3}{2}$.

**Check Your Progress**

**1.** Solve the equation $x^3-12x^2+39x-28=0$ the roots being in A.P.

Ans. Roots are 1, 4, 7.

2. Solve the equation $3x^3-26x^2+52x-24=0$, the roots being in G.P.

Ans. Roots are $\frac{2}{3}$, 2, 6.

**3.5.2 To find the condition that roots of given equation satisfy a given relation.**

**Example 4:** Find the condition that one root of the equation $px^3+qx^2+rx+s=0$ be equal to the sum of the other two.

**Solution:** The given equation is $px^3+qx^2+rx+s=0$ (1)

Let the roots of the equation be $\alpha,\beta,\gamma$ such that $\alpha=\beta+\gamma$

From (1), $\alpha+\beta+\gamma=-\frac{q}{p} \quad \Rightarrow \quad \alpha+\alpha=-\frac{q}{p} \quad \Rightarrow \alpha=-\frac{q}{2p}$

But $\alpha=-\frac{q}{2p}$ is a root of (1)

$$\therefore \qquad p\left(-\frac{q}{2p}\right)^3+q\left(-\frac{q}{2p}\right)^2+r\left(-\frac{q}{2p}\right)+s=0$$

or $\quad -q^3+2q^3-4pqr+8p^2s=0$

or $\quad q^3-4pqr+8p^2s=0$

which is the required condition.

**3.5.3 Common Roots of Two Equations**

Let $f_n(x)=0$ and $f_m(x)=0$ be two equations of degree n and m respectively.

Let $\alpha_1,\alpha_2,\alpha_3,.....,\alpha_r$ (r < n, r < m) be their common roots.

Then both $f_n(x)=0$ and $f_m(x)=0$ are divisible by $\varphi(x)=(x-\alpha_1)(x-\alpha_2)....(x-\alpha_r)$ which is, therefore, the H.C.F of $f_n(x)$ and $f_m(x)$.

Hence the common roots of the two equations such as $f_n(x)=0$ and $f_m(x)=0$ are given by $\varphi(x)=0$, where $\varphi(x)$ is H.C.F. of $f_n(x)$ and $f_m(x)$.

### 3.5.4 Equal or Multiple Roots of an Equation

A root $\alpha$ is said to be multiple root or repeated root of an equation $f(x)=0$ if $\alpha$ occurs more than once in the roots of the equation $f(x)=0$. If $\alpha$ is repeated m-times then $\alpha$ will be roots of $f'(x)=0$ repeated m-1 times, where $f'(x)=0$ is differential of $f(x)=0$ with respect to x.

**Rules to find multiple roots of an equation:**

**To find the multiple root of an equation** $f(x)=0$:

(i) Find $f'(x)$

(ii) Find the H.C.F of $f(x)$ and $f'(x)$.Let it be $\varphi(x)$.

(iii) Put $\varphi(x)=0$ and solve it.

(iv) If a root $\alpha$ of equation $\varphi(x)=0$ is repeated r times, then the same root $\alpha$ will be repeated r+1 times in the equation $f(x)=0$.

**Cor. 1.** If $\alpha$ is double root of the equation $f(x)=0$, then we have $f(\alpha)=0$ and $f'(\alpha)=0$. The required condition will be obtained by eliminating $\alpha$ from the equation $f(\alpha)=0$ and $f'(\alpha)=0$.

**Cor.2.** If $\alpha$ is triple root of the equation $f(x)=0$, then we have $f(\alpha)=0$, $f'(\alpha)=0$ and $f''(\alpha)=0$. The required condition will be obtained by eliminating $\alpha$ from the equation $f(\alpha)=0$, $f'(\alpha)=0$ and $f''(\alpha)=0$.

**Check Your Progress**

1. Solve the equation $3x^3-19x^2+33x-9=0$ which has repeated roots.

Ans.3, 3, $\frac{1}{3}$

2. Solve the equation $x^5-15x^3+10x^2+60x-72=0$ by testing for equal roots.

Ans. 2,2,2,-3 and -3.

## 3.6 TRANSFORMATION OF EQUATIONS

To transform an equation into another whose root shall be equal in magnitude but opposite in sign to those of the given equation.

Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ be the root of the equation

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0 \qquad (1)$$

We will find out an equation whose roots are $-\alpha_1, -\alpha_2, -\alpha_3 \ldots\ldots, -\alpha_n$.

If the new equation is in y then the functional relation between the roots of the equation is,

y=-x or x=-y

Putting the value of x in (1), we get $f(-y)=0$

$$a_0(-y)^n+a_1(-y)^{n-1}+a_2(-y)^{n-2}+....+a_{n-1}(-y)+a_n=0$$

$$a_0(-1)^n y^n+a_1(-1)^{n-1}y^{n-1}+a_2(-1)^{n-2}y^{n-2}+....+a_{n-1}(-1)y+a_n=0$$

Since n can be even as well as odd, we cannot conclude regarding the sign of $(-1)^n$, $(-1)^{n-1}$ etc.

So multiplying throughout by $(-1)^n$, we have

$$a_0(-1)^{2n}y^n + a_1(-1)^{2n-1}y^{n-1} + a_2(-1)^{2n-2}y^{n-2} + .... + a_{n-1}(-1)^{n+1}y + a_n(-1)^n = 0$$

But $(-1)^{2n} = 1$

$$(-1)^{2n-1} = (-1)^{2n}(-1)^{-1} = -1$$

……………………………………………………

$$(-1)^{n+1} = (-1)^{n-1}(-1)^2 = (-1)^{n-1}$$

Thus the required equation is

$$a_0y^n - a_1y^{n-1} + a_2y^{n-2} + .... + a_{n-1}(-1)^{n-1}y + a_n(-1)^n = 0$$

Expressing the equation in term of variable x, we get

$$a_0x^n - a_1x^{n-1} + a_2x^{n-2} + .... + a_{n-1}(-1)^{n-1}x + a_n(-1)^n = 0$$

Which is the required transformed equation.

**3.6.1 Root Multiplied by a Given Number:**

To transform an equation into another whose roots are m times those of the given equation.

Let $\alpha_1, \alpha_2, \alpha_3 ……, \alpha_n$ be the roots of the given equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + .... + a_{n-1}x + a_n = 0$$

It is required to find out an equation whose roots are m times the roots of one (1) i.e., we require an equation whose roots are $m\alpha_1, m\alpha_2, m\alpha_3 ……, m\alpha_n$.

If the new equation is in y, then the functional relation between the roots of the two equation is

$y = mx$ or $x = \dfrac{y}{m}$

Putting this value of x in (1), we get

$$f\left(\frac{y}{m}\right)=0$$

i.e., $a_0\left(\dfrac{y}{m}\right)^n + a_1\left(\dfrac{y}{m}\right)^{n-1} + a_2\left(\dfrac{y}{m}\right)^{n-2} + ..... + a_{n-1}\left(\dfrac{y}{m}\right) + a_n = 0$

Multiplying throughout by $m^n$ , we get

$$a_0y^n + a_1y^{n-1} + a_2y^{n-2} + ..... + a_{n-1}y + a_n = 0$$

Expressing it in x, we have

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ..... + a_{n-1}x + a_n = 0$$

Which is the required transformed equation.

**Note: 1.** In order to form an equation whose roots are m times the roots of the given equation, multiply the successive coefficients beginning with the coefficient of $x^{n-1}$ by $m, m^2, m^3 ...., m^n$ .respectively. If any power of x is missing it should be regarded as supplied with a zero coefficient.

2. If the roots of the given equation are to be divided by m, it means they are to be multiplied by $\dfrac{1}{m}$.

**3.6.2 Reciprocal Roots**

We will transform an equation whose roots are reciprocal of the roots of the given equation.

Consider an equation $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$ (i)

Let $\alpha_1,\alpha_2,\alpha_3......,\alpha_n$ be the roots of the given equation. We will find out an equation whose roots are $\frac{1}{\alpha_1},\frac{1}{\alpha_2},\frac{1}{\alpha_3},......,\frac{1}{\alpha_n}$

Let us suppose that the new equation formed will be in y. Then $\frac{y}{x}$ or $x=\frac{1}{y}$ is the functional relation between the roots of the two equations.

By putting this value of x in (1), we get

$$f(\frac{1}{y})=a_0\left(\frac{1}{y}\right)^n+a_1\left(\frac{1}{y}\right)^{n-1}+a_2\left(\frac{1}{y}\right)^{n-2}+....+a_{n-1}\left(\frac{1}{y}\right)+a_n=0$$

Multiplying throughout by $y^n$, we get

$$a_0+a_1y+a_2y^2+....+a_{n-1}y^{n-1}+a_ny^n=0$$

Expressing it in x, we have

$$a_0+a_1x+a_2x^2+....+a_{n-1}x^{n-1}+a_nx^n=0$$

**Example 5:** Find an equation whose roots are equal in magnitude but opposite in sign to the equation $x^5+11x^4+7x^3-16x^2-12x+15=0$.

Solution: We know that when we change x to –x, the signs of the coefficients of terms with odd powers of x change.

Thus, the required equation is $-x^5+11x^4-7x^3-16x^2+12x+15=0$

or $x^5-11x^4+7x^3+16x^2-12x-15=0$

**Example 6:** Find an equation whose roots are four times the roots of the equation $x^3++2x^2+3x-5=0$ .

Solution. The given equation is $x^3+2x^2+3x-5=0$.

Since the given equation is complete, therefore, multiplying the successive terms by $4^0,4^1,4^2,4^3$ respectively, we have

$$4^0.x^3+4^1.2x^2+4^2.3x-4^3.5=0$$

or $x^3+8x^2+48x-320=0$.

Which is the required equation.

**Example 7:** Remove the fractional coefficients from the equation

$$x^4+\frac{1}{2}x^3-\frac{5}{3}x^2+\frac{2}{3}x-1=0$$

**Solution.** To remove the fractional coefficients, we multiply the roots of equation (1) by 6 which is the L.C.M of 2 and 3, the denominator of the fractional coefficients, which are prime to each other.

Thus, the transformed equation is $x^4+6.\frac{1}{2}x^3-6^2.\frac{5}{3}x^2+6^3.\frac{2}{3}x-6^4.1=0$

or $x^4+3x^3-60x^2+144x-1296=0$.

**Example 8:** Transform the equation into one whose roots are twice the reciprocals of the roots of the equation $x^4+3x^3-6x^2+2x-4=0$

Solution. The given equation is $x^4+3x^3-6x^2+2x-4=0$

Replacing x by $\frac{2}{x}$, we get

$$\left(\frac{2}{x}\right)^4+3\left(\frac{2}{x}\right)^3-6\left(\frac{2}{x}\right)^2+2\left(\frac{2}{x}\right)-4=0$$

or $\quad 16+24x-24x^2+4x^3-4x^4=0$

or $\quad x^4-x^3+6x^2-6x-4=0$

Which is the required equation.

**Example 9:** Find the condition that the roots of the cubic $x^3-px^2+qx-r=0$ may be in H.P . Hence or otherwise solve the equation $6x^3-11x^2+6x-1=0$

Solution. (i) The given equation is $x^3-px^2+qx-r=0$

It is given that its roots are in H.P.

Changing x to $\frac{1}{y}$ in (1), we get $\left(\frac{1}{y}\right)^3-p\left(\frac{1}{y}\right)^2+q\left(\frac{1}{y}\right)-r=0$

i.e. $\quad ry^3-qy^2+py-1=0$

Clearly the roots of (2) are in A.P. Let the roots of (20 be a – d, a, a + d.

Then sum of the roots = 3a=$\frac{q}{r}\Rightarrow a=\frac{q}{3r}$

$\Rightarrow \quad y=\frac{q}{3r}$ is one root of (2)

$$\Rightarrow \quad r\left(\frac{q}{3r}\right)^3-q\left(\frac{q}{3r}\right)^2+p\left(\frac{q}{3r}\right)-1=0$$

or $\quad \frac{q^3}{27r^2}-\frac{q^3}{9r^2}+\frac{pq}{3r}-1=0$ i.e., $q^3-3q^3+9pqr-27r^2=0$

$\therefore$ Required condition is $2q^3-9pqr+27r^2=0$

The given equation is $6x^3-11x^2+6x-1=0$

Dividing by 6, we get $x^3-\frac{11}{6}x^2+x-\frac{1}{6}=0$

Comparing (1) and (3), we get $p=\frac{11}{6}$, $q=1$ and $r=6$

L.H.S of (3)= $2q^3-9pqr+27r^2$

$$=2-9\left(\frac{11}{6}\right)(1)\left(\frac{1}{6}\right)+27\left(\frac{1}{36}\right)^2=0$$

This shows that equation (1) and (4) satisfy the same conditions.

$\therefore \quad x=\frac{1}{y}=\frac{3r}{q}=\frac{1}{2}$ is one root of equation (4)

$\therefore \quad x-\frac{1}{2}$ is a factor of L.H.S. of (4).

By synthetic division,

$$\begin{array}{c|cccc} \frac{1}{2} & 6 & -11 & 6 & -1 \\ & \ldots & -3 & -4 & 1 \\ \hline & 6 & -8 & 2 & 0 \end{array}$$

$\therefore$ Depressed equation is $6x^2-8x+2=0$

i.e. $3x^2-4x+1=0$

or $(3x-1)(x-1)=0 \Rightarrow x=\frac{1}{3},1$

Hence the roots of (1) are $1,\frac{1}{2},\frac{1}{3}$.

**Check Your Progress**

1. Remove the fractional coefficient from the equation $x^4+\frac{1}{2}x^3-\frac{5}{3}x^2+\frac{2}{3}x-1=0$

**Ans.** $x^4-24x^2+65x-55=0$.

2. Solve the equation $40x^4-22x^3-21x^2+2x+1=0$ if the roots are in H.P.

Ans. $-\frac{1}{5},-\frac{1}{2},1,\frac{1}{4}$.

**3.6.3 Root Diminished by a Given Number:**

We will transform an equation in to another equation whose roots will be the roots of the given equation diminished by h.

Consider an equation $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0 \qquad$ (i)

Let $\alpha_1,\alpha_2,\alpha_3\ldots\ldots,\alpha_n$ be the roots of the given equation. We will find out an equation whose roots are $\alpha_1-h,\alpha_2-h,\alpha_3-h,\ldots\ldots,\alpha_n-h$.

Let the new equation formed will be in y. Then y= x- h or x = y + h is the functional relation between the roots of two equations.

By putting the value of x in (i), we get

$f(y+h)=a_0(y+h)^n+a_1(y+h)^{n-1}+a_2(y+h)^{n-2}+....+a_{n-1}(y+h)+a_n=0$

Simplifying and arranging in descending power of y, we get

$A_0y^n+A_1y^{n-1}+A_2y^{n-2}+....+A_{n-1}y+A_n=0$

Where $A_0, A_1,....., A_n$ are constants to be determined. ... (ii) Putting y = x – h in (ii)

$\therefore f(x)=A_0(x-h)^n+A_1(x-h)^{n-1}+A_2(x-h)^{n-2}+....+A_{n-1}(x-h)+A_n=0$

L.H.S of this equation is identical with the L.H.S. of equation (i).

This shows that if f(x) is divided by (x-h), then the remainder is $A_n$ and the quotient is

$A_0(x-h)^{n-1} + A_1(x-h)^{n-2} + A_2(x-h)^{n-3} + .... + A_{n-1} = 0$

If we divide this quotient by (x-h), then we get $A_{n-1}$ as the remainder and the second quotient is

$A_0(x-h)^{n-2} + A_1(x-h)^{n-3} + A_2(x-h)^{n-4} + .... + A_{n-2} = 0$

Repeating this process n time, the nth remainder will be $A_1$ and the nth quotient will be $A_0$

By equating the coefficients of $x^n$ in (i) and (iii), we get $A_0 = a_0$.

Thus, the division of f(x) successively by (x-h) we get $A_n, A_{n-1}, ....., A_1, A_0$ as the successive remainders and $A_0 (= a_0)$ as the last quotient. Then the transformed equation in x is $A_0x^n + A_1x^{n-1} + A_2x^{n-2} + .... + A_{n-1}x + A_n = 0$.

**2.6.4 Removal of terms in General**

We will remove a particular term from the transformed equation by decreasing the roots of the given equation by a suitable number.

Consider an equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + .... + a_{n-1}x + a_n = 0$ .......(i)

Suppose the roots of the equation (i) are diminished by h.

Let the new equation formed will be in y. Then y= x – h or x = y + h is the functional relation between the roots of two equations.

By putting the value of x in (i), we get

$f(y+h) = a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + .... + a_{n-1}(y+h) + a_n = 0$

By applying binomial theorem and arranging the terms in the descending power of y, we get

$$a_0y^n + (na_0h + a_1)y^{n-1} + \left[\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_1h + a_2\right]y^{n-2} + ...$$
$$+ (a_0h^n + a_1h^{n-1} + .. + a_{n-1}h + a_n)$$

Now to remove any particular term we equate the coefficient of that term to zero and the values of h thus obtained will be the required numbers by which the roots are to be diminished to get the required transformed equation.

Note: (i) To remove the second term: $na_0h + a_1$ or $h = -\frac{a_1}{na_0}$

(ii) To remove the third term: $\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_1h + a_2 = 0$

(iii) To remove the fourth term, solve a cubic in h. To remove the last term, solve the equation $f(h) = 0$.

**3.6.5 Transformation of Cubic**

We will reduce the cubic $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form in which second term is missing and the coefficient of the leading term is unity, all other coefficients being integers. And we will find the relation between the roots of the transformed equation and the given equation.

Consider an equation $f(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ .....(i)

Suppose the transformed equation is in y. Then y = x – h or y=x+h

Let us diminish the roots of (i) by h by putting x = y + h

$f(y+h) = a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$

Now, arranging the term in descending power of y, we get

$$a_0y^3+3(a_0h+a_1)y^2+3(a_0h^2+2a_1h+a_2)y+(a_0h^3+3a_1h^2+3a_2h+a_3)=0$$

Let us write this equation as

$$A_0y^3+3A_1y^2+3A_2y+A_3=0 \quad ....(ii) \qquad [\because A_0=a_0]$$

For removing the second term, take $a_0h+a_1=0$ or $h=-\dfrac{a_1}{a_0}$

We have $A_2=a_0h^2+2a_1h+a_2=a_0\left(-\dfrac{a_1}{a_0}\right)^2+2a_1\left(-\dfrac{a_1}{a_0}\right)+a_2 \qquad \left[\because h=\left(-\dfrac{a_1}{a_0}\right)\right]$

$=\dfrac{a_0a_2-a_1^{\ 2}}{a_0}=\dfrac{H}{a_0}$ Where H=$a_0a_2-a_1^{\ 2}$

Also, $A_3=a_0h^3+3a_1h^2+3a_2h+a_3=a_0\left(-\dfrac{a_1}{a_0}\right)^3+3a_1\left(-\dfrac{a_1}{a_0}\right)^2+3a_2\left(-\dfrac{a_1}{a_0}\right)+a_3$

$=\dfrac{a_0^{\ 2}a_3-3a_0a_1a_2+2a_1^{\ 3}}{a_0^{\ 2}}=\dfrac{G}{a_0^{\ 2}}$, where G=$a_0^{\ 2}a_3-3a_0a_1a_2+2a_1^{\ 3}$

Now putting this value in (ii), we get

$$a_0y^3+\frac{3H}{a_0}y+\frac{G}{a_0^{\ 2}}=0$$

To make the coefficients of the leading term as unity, we divide this equation by $a_0$

$$\Rightarrow a_0y^3+\frac{3H}{a_0^{\ 2}}y+\frac{G}{a_0^{\ 3}}=0$$

Multiplying roots by $a_0$, we get $Z^3+3HZ+G=0$ where Z=$a_0y$

This is the required transformed equation. If $\alpha^{'}, \beta^{'}$ and $\gamma^{'}$ be the roots of this transformed equation. Then

Z=$a_0y=a_0(x-h)$ $\qquad$ [$\because$ Z=$a_0y$ and $y=(x-h)$]

$=a_0\left(x+\dfrac{a_1}{a_0}\right)=a_0x+a_1 \qquad [\because h=-\dfrac{a_1}{a_0}]$

$\Rightarrow \alpha^{'}=a_0\alpha+a_1; \beta^{'}=a_0\beta+a_1; \gamma^{'}=a_0\gamma+a_1$, where $\alpha, \beta$ and $\gamma$ are the roots of equation (i).

From (i), we have sum of the roots = $\alpha+\beta+\gamma=-\dfrac{3a_1}{a_0}$

$$\therefore \alpha^{'}=a_0\left(\alpha+\frac{a_1}{a_0}\right)=a_0\left(\alpha-\frac{\alpha+\beta+\gamma}{3}\right)=\frac{a_0}{3}(2\alpha-\beta-\gamma)$$

$\therefore \beta^{'}=\dfrac{a_0}{3}(2\beta-\alpha-\gamma)$ and $\gamma^{'}=\dfrac{a_0}{3}(2\gamma-\beta-\alpha)$.

**3.6.6 Transformation of Biquardatic**

We will reduce the cubic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ to the form in which second term is missing and the coefficient of the leading term is unity, all other coefficients being integers. And we will find the relation between the roots of the transformed equation and the given equation.

Consider an equation $f(x)=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ .....(i)

Suppose the transformed equation is in y. Then y = x - h or x = y + h
Let us diminish the roots of (i) by h by putting x = y + h

$$f(y+h) = a_0(y+h)^4 + 4a_1(y+h)^3 + 6a_2(y+h)^2 + 4a_3(y+h) + a_4 = 0$$

Now, arranging the term in descending power of y, we get

$$a_0 y^4 + 4(a_0 h + a_1)y^3 + 6(a_0 h^2 + 2a_1 h + a_2)y^2 + 4(a_0 h^3 + 3a_1 h^2 + 3a_2 h + a_3)y$$
$$+ (a_0 h^4 + 4a_1 h^3 + 6a_2 h^2 + 4a_3 h + a_4) = 0$$

Let us write this equation as

$$A_0 y^4 + 4A_1 y^3 + 6A_2 y^2 + 4A_3 y + A_4 = 0 \quad ....\text{(ii)} \qquad [\because A_0 = a_0]$$

For removing the second term, take $a_0 h + a_1$ or $h = -\dfrac{a_1}{a_0}$

We have $A_2 = a_0 h^2 + 2a_1 h + a_2 = a_0\left(-\dfrac{a_1}{a_0}\right)^2 + 2a_1\left(-\dfrac{a_1}{a_0}\right) + a_2 \qquad \left[\because h = \left(-\dfrac{a_1}{a_0}\right)\right]$

$= \dfrac{a_0 a_2 - a_1^2}{a_0} = \dfrac{H}{a_0}$ Where H$=a_0 a_2 - a_1^2$

Also, $A_3 = a_0 h^3 + 3a_1 h^2 + 3a_2 h + a_3 = a_0\left(-\dfrac{a_1}{a_0}\right)^3 + 3a_1\left(-\dfrac{a_1}{a_0}\right)^2 + 3a_2\left(-\dfrac{a_1}{a_0}\right) + a_3$

$= \dfrac{a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3}{a_0^2} = \dfrac{G}{a_0^2}$, where G$=a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3$

and $A_4 = a_0 h^4 + 4a_1 h^3 + 6a_2 h^2 + 4a_3 h + a_4 = a_0\left(-\dfrac{a_1}{a_0}\right)^4 + 4a_1\left(-\dfrac{a_1}{a_0}\right)^3 + 6a_2\left(-\dfrac{a_1}{a_0}\right)^2 + 4a_3\left(-\dfrac{a_1}{a_0}\right) + a_4$

$= \dfrac{a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4}{a_0^3}$

$= \dfrac{a_0^2(a_0 a_4 - 4a_1 a_3 + 3a_2^2) - 3(a_0 a_2 - a_1^2)^2}{a_0^3} = \dfrac{a_0 \text{I} - 3\text{H}^2}{a_0^3}$

where $\text{I} = a_0 a_4 - 4a_1 a_3 + 3a_2^2$

Now putting these values in (ii), we get

$$a_0 y^4 + \frac{6H}{a_0} y^2 + 4\frac{G}{a_0^2} y + \frac{a_0 \text{I} - 3\text{H}^2}{a_0^3} = 0$$

$$\Rightarrow y^4 + \frac{6H}{a_0^2} y^2 + 4\frac{G}{a_0^3} y + \frac{a_0 \text{I} - 3\text{H}^2}{a_0^4} = 0$$

Multiplying roots by $a_0$, we get $Z^4 + 6HZ^2 + 4GZ + (a_0 \text{I} - 3\text{H}^2) = 0$ where $Z = a_0 y$

This is the required transformed equation. If $\alpha', \beta', \gamma'$ and $\delta'$ be the roots of this transformed equation. Then $Z = a_0 y = a_0(x-h) \qquad [\because Z = a_0 y$ and $y = (x-h)]$

$= a_0\left(x + \dfrac{a_1}{a_0}\right) = a_0 x + a_1 \qquad [\because h = -\dfrac{a_1}{a_0}]$

$\Rightarrow \alpha' = a_0\alpha + a_1$; $\beta' = a_0\beta + a_1$; $\gamma' = a_0\gamma + a_1$ and $\delta' = a_0\delta + a_1$, where $\alpha, \beta, \gamma$ and $\delta$ are the roots of equation (i).

From (i), we have sum of the roots = $\alpha + \beta + \gamma + \delta = -\dfrac{4a_1}{a_0}$

$$\therefore \alpha' = a_0\left(\alpha + \frac{a_1}{a_0}\right) = a_0\left(\alpha - \frac{\alpha+\beta+\gamma+\delta}{4}\right) = \frac{a_0}{4}(3\alpha - \beta - \gamma - \delta)$$

$$\therefore \beta' = \frac{a_0}{4}(3\beta - \alpha - \gamma - \delta),\ \gamma' = \frac{a_0}{4}(3\gamma - \beta - \alpha - \delta) \text{ and } \delta' = \frac{a_0}{4}(3\delta - \beta - \alpha - \gamma).$$

### 3.6.7 Removal of Second and Third Term

Consider an equation $f(x) = a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ .....(i)

Suppose the transformed equation is in y. Then y = x - h or x = y + h

Let us diminish the roots of (i) by h by putting x = y + h

$$f(y+h) = a_0(y+h)^3 + 3a_1(y+h)^2 + 3a_2(y+h) + a_3 = 0$$

Now, arranging the term in descending power of y, we get

$$a_0y^3 + 3(a_0h + a_1)y^2 + 3(a_0h^2 + 2a_1h + a_2)y + (a_0h^3 + 3a_1h^2 + 3a_2h + a_3) = 0$$

The second and third terms will be removed simultaneously if $a_0h + a_1 = 0$ or $h = -\dfrac{a_1}{a_0}$ and

$$a_0h^2 + 2a_1h + a_2 = 0$$

Putting the value of h in (ii), we get

$$a_0\left(-\frac{a_1}{a_0}\right)^2 + 2a_1\left(-\frac{a_1}{a_0}\right) + a_2 = 0 \Rightarrow \frac{a_1^2}{a_0} - 2\frac{a_1^2}{a_0} + a_2 = 0$$

$$\Rightarrow a_0a_2 - a_1^2 = 0$$

$\Rightarrow$ H=0 is the required condition.

**Example 1:** Diminish the roots of $2x^5 - x^3 + 10x - 8 = 0$ by 5.

**Solution.** The given equation is $2x^5 - x^3 + 10x - 8 = 0$

In order to diminish the roots by 5 we have to divide the given equation successively by (x-5)

Thus h (the multiplier) = 5.

By successive application of synthetic division, we have

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 2 | 0 | -1 | 0 | 10 | -18 |
| | ... | 10 | 50 | 245 | 1225 | 6175 |
| | 2 | 10 | 49 | 245 | 1235 | **6167** |
| | ... | 10 | 100 | 745 | 4950 | |
| | 2 | 20 | 149 | 990 | **6185** | |
| | .... | 10 | 150 | 1495 | | |
| | 2 | 30 | 299 | **2485** | | |
| | … | 10 | 200 | | | |
| | 2 | 40 | **499** | | | |
| | ... | 10 | | | | |
| | 2 | **50** | | | | |
| | ... | **2** | | | | |

**Check Your Progress**

1. Diminish the roots of equation $x^4 - 5x^3 + 7x^2 - 17x + 11 = 0$, by 4.

**3.6.8 Transformation in General**

Consider an equation $f(x) = 0$

and let $\alpha_1, \alpha_2, ...., \alpha_n$ be its roots.

We will find out an equation whose roots are $\phi(\alpha_1), \phi(\alpha_2), ...., \phi(\alpha_n)$ where $\phi(x)$ is an algebraic function of x. Let us suppose that the new equation formed will be in y. Then the functional relation between the roots of the two equation is

$$y = \phi(x) \qquad ......(ii)$$

Now, we eliminate x from (i) and (ii) and get an equation F(y) = 0 which is the required transformed equation.

**Example 2:** Find the equation whose roots are the squares of the roots of $x^3 + qx + r = 0$.

Solution. The given equation is $x^3 + qx + r = 0$ (1)

Let its roots be $\alpha, \beta, \gamma$

It is required to find out an equation whose roots are $\alpha^2, \beta^2, \gamma^2$.

If the new equation is in y, then $y = x^2$ (2)

Eliminating x between (1) and (2), we get

$$xy + xq + r = 0$$

or $x(y+q) = -r$ [Transposing]

or $x^2(y+q)^2 = r^2$ [Squaring]

or $y(y+q)^2 = r^2$ [$\because x^2 = y$]

or $y(y^2 + q^2 + 2yq) = r^2$

or $y^3 + 2yq + q^2y - r^2 = 0$

Which is the required equation.

**Example 3:** If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$; form an equation whose roots are $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$.

Solution. Since $\alpha, \beta, \gamma$ are the roots of the equation

$$x^3 + px^2 + qx + r = 0; \qquad ..... (1)$$

$$\alpha + \beta + \gamma = -p \qquad ..... (2)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q \qquad ..... (3)$$

$$\alpha\beta\gamma = -r \qquad ...... (4)$$

If the transformed equation is in y, then

$$y = \alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha - \frac{\alpha}{-r} \qquad \text{[From (4)]}$$

$= \alpha + \dfrac{\alpha}{r} = x + \dfrac{x}{r}$ [$\because \alpha$ is a root of (1) ]

$= x\left(\dfrac{r+1}{r}\right)$ i.e. $x = \dfrac{ry}{r+1}$

Putting this value of x in (1), we get

$$\frac{r^3y^3}{(r+1)^3} + p.\frac{r^2y^2}{(r+1)^2} + q.\frac{ry}{r+1} + r = 0$$

or $\quad r^2y^3 + pry^2(r+1) + qy(r+1)^2 + (r+1)^3 = 0$

or $\quad r^2y^3 + pr(r+1)y^2 + q(r+1)^2y + (r+1)^3 = 0$

Which is the required transformed equation.

**Example 4:** Form an equation whose roots shall be the squares of the roots of the equation $x^3 + 3x^2 + 6x + 1 = 0$.

Solution: The given equation is $x^3 + 3x^2 + 6x + 1 = 0$ (i)

Let the new equation be in y. Then $y = x^2$ (ii)

From (i) and (ii), $xy + 3y + 6x + 1 = 0$

$\therefore \quad x(y+6) = (1+3y)$

By squaring both sides, we get

$x^2(y+6)^2 = (1+3y)^2 \Rightarrow y(y^2 + 36 + 12y) = (9y^2 + 1 + 6y)$

$y^3 + 3y^2 + 30y - 1 = 0$, which is the required equation.

**3.6.9 Equation of Squared Difference of a Cubic**

**Equation whose roots are the squares of the differences of the roots of the equation** $x^3+qx+r=0$

Consider the cubic equation $x^3+qx+r=0$ (i)

Let $\alpha, \beta, \gamma$ are the roots of the equation

$x^3 + qx + r = 0;$ ..... (1)

$\alpha + \beta + \gamma = 0$ ...... (2)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ....... (3)

$\alpha\beta\gamma = -r$ ....... (4)

We have to find an equation whose roots are $(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$

If the transformed equation is in y, then

$y = (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$= [(\alpha+\beta+\gamma) - \alpha]^2 - 4\dfrac{\alpha\beta\gamma}{\gamma}$

$= [0-\alpha]^2 - 4\dfrac{r}{\alpha}$ [$\because \alpha+\beta+\gamma = 0; \alpha\beta\gamma = -r$]

$\Rightarrow y = x^2 + \dfrac{4r}{x}$ [$\because$ α is a root of (i)]

$\Rightarrow x^3 - xy + 4r = 0$ ....((ii)

Subtracting (ii) from (i), we get

$x(q+y) - 3r = 0 \Rightarrow x = \dfrac{3r}{q+y}$ ....(iii)

Putting this value of x in (i), we get

$$\left(\frac{3r}{q+y}\right)^3 - \left(\frac{3r}{q+y}\right)y + 4r = 0$$

$\Rightarrow 27r^3 + 3qr(q+y)^2 + r(q+y)^3 = 0$

$\Rightarrow (q+y)^3 + 3q(q+y)^2 + 27r^2 = 0$

$\Rightarrow y^3 + 6qy^2 + 9q^2y + (4q^3 + 27r^2) = 0$

which is the required equation.

**Equation whose Roots are Squares of the Differences of the Roots of the Cubic**

$$a_0x^2 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

Consider the cubic equation $a_0x^2 + 3a_1x^2 + 3a_2x + a_3 = 0$ ... (i)

and let $x_1, x_2$ and $x_3$ be the roots of this equation.

We have to find an equation whose roots are $(x_1 - x_2)^2, (x_2 - x_3)^2$ and $(x_3 - x_1)^2$

By putting $y = a_0x + a_1$, $H = a_0a_2 - a_1^2$ and $G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$ equation (i) reduces to

$y^3 + 3Hy + G = 0$ .... (ii)

Let the roots of (ii) be $\alpha, \beta$ and $\gamma$.

$\therefore\ \alpha = a_0x_1 + a_1;\quad \beta = a_0x_2 + a_1;\quad \gamma = a_0x_3 + a_1$

$\Rightarrow \alpha - \beta = a_0(x_1 - x_2)$

$\Rightarrow \beta - \gamma = a_0(x_2 - x_3)$

$\Rightarrow \gamma - \alpha = a_0(x_3 - x_1)$

Let the new equation be in Z. Then $Z = (\alpha - \beta) = (\alpha + \beta)^2 - 4\alpha\beta$

$= [(\alpha + \beta + \gamma) - \gamma]^2 - \dfrac{4\alpha\beta\gamma}{\gamma} = (-\gamma)^2 + \dfrac{4G}{\gamma}$ [By (ii), $\alpha + \beta + \gamma = 0$ and $\alpha\beta\gamma = -G$ ]

$= \gamma^2 + \dfrac{4G}{\gamma} = y^2 + \dfrac{4G}{y}$ [$\because$ γ is a root of (ii)]

$\Rightarrow y^3 - Zy + 4G = 0$ ...... (iii)

Subtracting (iii) from (ii), we get

$(3H+Z)y - 3G = 0 \Rightarrow y = \dfrac{3G}{3H+Z}$

Putting this value of y in (ii), $\left(\dfrac{3G}{3H+Z}\right)^3 + 3H\left(\dfrac{3G}{3H+Z}\right) + G = 0$

$\Rightarrow (3H+Z)^3 + 9H(3H+Z)^2 + 27G^2 = 0$

$\Rightarrow Z^3+18HZ^2+81H^2Z+27(G^2+4H^3)=0$ ..... (iv)

The roots of this equation are $(\alpha-\beta)^2, (\beta-\gamma)^2$ and $(\gamma-\alpha)^2$

i.e. $a_0^{\ 2}(x_1-x_2)^2, a_0^{\ 2}(x_2-x_3)^2$ and $a_0^{\ 2}(x_3-x_1)^2$

If the new equation is in t, then $t=(x_1-x_2)^2=\frac{1}{a_0^{\ 2}}a_0^{\ 2}(x_1-x_2)^2=\frac{1}{a_0^{\ 2}}Z$

$\Rightarrow Z=a_0^{\ 2}t$

By putting this value of Z in (iv), we get

$a_0^{\ 6}t^3+18a_0^{\ 4}Ht^2+81a_0^{\ 2}H^2t+27(G^2+4H^3)=0$

Which is the required equation.

**Example 5:** Find the equation of squared difference of the roots of the equation $x^3-7x+6=0$.

Solution: The given equation $x^3-7x+6=0$ .... (i)

Let $\alpha, \beta, \gamma$ be the roots of the equation

$\therefore \quad \alpha+\beta+\gamma=0$

$\alpha\beta\gamma=-6$

Now we have to form an equation whose roots are $(\alpha-\beta)^2,(\beta-\gamma)^2,(\gamma-\alpha)^2$

If the new equation is in y, then $y=(\beta-\gamma)^2$

or $\quad y=(\beta+\alpha)^2-4\beta\gamma$

$$=(0-\alpha)^2-4(-\frac{6}{\alpha})=\alpha^2+\frac{24}{\alpha}=\frac{\alpha^3+24}{\alpha}$$

or $\quad y=\frac{x^3+24}{x}$ [Replacing $\alpha$ by x]

or $\quad xy=x^3+24$

or $\quad x^3-xy+24=0$ ....(ii)

To eliminate x between (i) and (ii), subtracting (ii) from (i), we have

$$-7x+xy-18=0$$

or $\quad x(y-7)=18\Rightarrow \quad x=\frac{18}{y-7}$

Putting this value of x in (1), we have

$$\left(\frac{18}{y-7}\right)^3-7\left(\frac{18}{y-7}\right)+6=0$$

or $\quad 5832-7(18)(y-7)^2+6(y-7)^3=0$

or $\quad 5832-126[y^2-14y+49]+6[y^3-343-21y^2+147y]=0$

or $\quad 5832-126y^2+1764y-6174+6y^3-2058-126y^2+882y=0$

or $\quad 6y^3-252y^2+2646y-2400=0$

or $\quad y^3-42y^2+441y-400=0$.

## 3.7 SUMMARY

- If $f(x)$ and $g(x)$ are two nonzero polynomial, then there exist unique polynomial $q(x)$ and $r(x)$ such that $f(x)=q(x).g(x)+r(x)$, where $r(x)$ is either a zero polynomial or degree of $r(x) <$ degree of $g(x)$. Where $q(x)$ is called quotient and $r(x)$ is called remainder when $f(x)$ is divided by $g(x)$.
- If a polynomial $f(x)$ is divided by x-a, then the remainder is equal to $f(a)$.
- If h is a root of the equation f(x)=0, then (x-h) is a factor of f(x) and conversely.
- To transform an equation into another whose root shall be equal in magnitude but opposite in sign, change signs of coefficients of the terms with odd powers of x in the given equation.
- In order to form an equation whose roots are m times the roots of the given equation, multiply the successive coefficients beginning with the coefficient of $x^{n-1}$ by m, $\mathrm{m}^2$, $\mathrm{m}^3$ ...., $\mathrm{m}^n$ .respectively. If any power of x is missing it should be regarded as supplied with a zero coefficient.
- We transform an equation $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n=0$ in to another equation whose roots will be the roots of the given equation diminished by h. Then the transformed equation is $\mathrm{A}_0x^n+\mathrm{A}_1x^{n-1}+\mathrm{A}_2x^{n-2}+....+\mathrm{A}_{n-1}x+\mathrm{A}_n=0$ ,where $\mathrm{A}_0=a_0$, $\mathrm{A}_n, \mathrm{A}_{n-1}$,....., $\mathrm{A}_1$ are constants.

## 3.8 KEY TERMS

- **Real Polynomial:** A polynomial $f(x)=a_0x^n+a_1x^{n-1}+...+a_n$ is said to be real polynomial If all the coefficients $a_0, a_1,..., a_n$ in $f(x)$ are real.
- **Zero Polynomial:** If all the coefficients $a_0, a_1,..., a_n$ in $f(x)$ are zero then polynomial is said to be **zero polynomial**.
- **Complete and Incomplete equation:**A General Equation of degree n is said to be complete if it contains all power of the variable x from 0 to n. For example $a_0x^3+a_1x^2-a_3x+a_4=0$ is **complete equation** of third degree.
- **Root of an Equation:**The value of x for which $f(x)$ vanishes is called root of equation $f(x)=0$ .For example if $f(h)=0$ , then h is called root of the equation $f(x)=0$ .

## 3.9. QUESTIONS AND EXERCISES

1. Given that -6 is a root of equation $x^3+2x^2-17x+42=0$, solve it.
2. From an equation with rational coefficients two of whose roots are $1+5i$ and 5-i .
3. Show that the equation $\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+.......+\frac{H^2}{x-h}=K$ has all real roots.
4. Find an equation of the lowest degree with real coefficients which has 1+2i and 3-I as two of its roots.
5. Solve the equation $x^4+4x^3+6x^2+4x+5=0$ given that one root is i.
6. If $\alpha,\beta,\gamma$ are the roots of the equation $(a-x)(b-x)(c-x)+1=0$, then prove that a,b,c are the roots of the equation $(x-\alpha)(x-\beta)(x-\gamma)+1=0$ .
7. Solve the equation $4x^3-4x^2-15x+18=0$, two of its roots being equal.
8. Solve the equation $3x^4-40x^3+130x^2-120x+27=0$ given that the product of two roots is equal to the product of the other two roots.
9. Solve the equation $x^4+x^3-16x^2-4x+48=0$, having given that the product of two of the roots is 6.

10. Find the condition that the roots of the equation $x^3 + px^2 + qx + r = 0$ should have two roots $\alpha, \beta$, connected by the relation $\alpha\beta + 1 = 0$.

11. Prove that for two roots of the cubic $x^3 + ax + b = 0$ to be equal, the condition is $\frac{b^2}{4} + \frac{a^3}{27} = 0$.

12. If $\alpha, \beta, \gamma, \delta$ are the roots of the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$, find the condition that the root should be connected by the relation $\beta + \gamma = \alpha + \delta$ and hence solve the equation $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ .

13. Without solving the equation $x^3 - x^2 - 2x + 2 = 0$, prove that it has no multiple roots.

14. Solve the equation $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0$, by testing for equal roots.

15. Solve the equation $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$, if the roots are given to be in H.P.

16. The difference between two roots of the equation $2x^3 + x^2 - 7x - 6 = 0$ is 3. Solve it by diminishing the roots by 3.

17. Remove the second term from the equation $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

18. The difference of two roots of the equation $x^3 - 13x^2 + 15x + 189 = 0$ is 2. Solve it by increasing the roots by 2.

19. Transform the equation $x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$ into one which is wanting in the third term.

20. If $\alpha, \beta, \gamma$ are the roots of the cubic $x^3 + ax^2 + bx + c = 0$ find the equation whose roots are $\frac{\alpha}{\beta + \gamma}, \frac{\beta}{\alpha + \gamma}, \frac{\gamma}{\alpha + \beta}$.

21. If the roots of equation $x^3 - 6x^2 + 11x - 6 = 0$ be $\alpha, \beta, \gamma$, find the equation whose roots are $\beta^2 + \gamma^2, \gamma^2 + \alpha^2, \alpha^2 + \beta^2$.

## 3.10 FURTHER READING

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**UNIT – IV**

# SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS

**STRUCTURE**



## 4.0 INTRODUCTION

Descarte's rule of signs was first described by Rene Descartes. It is a method for determining the number of positive or negative real roots of a polynomial. This rule gives us an upper bound number of positive or negative roots of a polynomial. It is not a deterministic rule i.e., It does not tell the exact number of positive or negative roots. , we are going to learn the solution of cubic and biquadratic equations.

## 4.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Find out the maximum number of positive and negative roots of a complete equation.
- Find out the maximum number of positive, negative and imaginary roots of an incomplete equation.
- Solve cubic equation by Cardan's Method.
- Solve biquadratic equation by Descarte's and Ferrari's method.

## 4.2 DESCARTE'S RULES OF SIGNS

Here we will discuss the definition of some concept related to Descate's rule of sign.

**Continuation or Permanence of sign**

A continuation or permanence of sign is said to occur in a polynomial $f(x)$, whose terms are arranged in descending power of x, if the two successive terms have the same sign.

For example: In $x^7+x^6-7x^5-9x^4-7x^3+4x^2-11x+10$, there are three continuation of sign occurring at $+x^6$, $-9x^4$, $-7x^3$.

For example: In $x^6-7x^5+11x^4-5x^3-6x^2-19x+11$, there are four changes or variations of signs occurring at $-7x^5$, $11x^4$, $-5x^3$ and 11.

**Complete equation**

An equation:

(i) The degree of the equation is equal to the sum of the number of continuation and variations of signs.

(ii) A continuation of sign becomes a variation of sign, if x is changed into –x and vice a versa.

**Ambiguity of Sign**

When any term of a polynomial $f(x)$ has the double sign $\pm$, $\mp$, the ambiguity of sign is said to occur.

**Rule 1:** The number of changes of signs of the coefficients in $f(x)$ always exceed the number of positive roots in any polynomial equation $f(x)=0$ with real coefficients.

**Rule 2:** The number of changes of signs of the coefficients in $f(-x)$ always exceed the number of negative roots in any polynomial equation $f(x)=0$ with real coefficients.

**Proof:** Let the sign of terms in a polynomial $f(x)$ be

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| + | + | - | - | - | + | - | + | - |

There are five changes of signs in the given polynomial. On multiplying the given polynomial by a binomial x-h (h $>$ 0) in which the signs are + -, we get

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| + | + | - | - | - | + | - | + | - | | |
| + | - | | | | | | | | | |
| + | + | - | - | - | + | - | + | - | | |
| | - | - | + | + | + | - | + | - | + | |
| + | $\pm$ | - | $\mp$ | $\mp$ | + | - | + | - | + | ... (i) |

We have three ambiguous signs in the resulting polynomial which replace each continuation of sign in the original polynomial.

Now, consider the most unfavorable case in which continuation replace each ambiguity.

It can be seen that the resulting series of signs is same as the original with an additional change of sign at end.

Now, if we take the lower signs of equation (i), we get

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| + | - | - | + | + | + | - | + | - | + |

Thus, we can see that in both cases there are six changes of signs which is at least one more than the number of changes of signs in the original polynomial.

Hence on multiplying a polynomial by a factor of the form x-h (h > 0), at least one additional change of signs will always introduced.

Now, let an nth degree equation $f(x)=0$ has p positive roots (p < n) say, $h_1, h_2, ......., h_p$ and remaining roots are negative, zero or imaginary.

Then, $f(x)=(x-h_1)(x-h_2).......(x-h_p)\phi(x)$, where $\phi(x)$ is of (n - p)th degree.

The expression $\phi(x)$ may or may not have any change of sign, but when it is multiplied by $(x-h_1)(x-h_2).......(x-h_p)$ then at least p new changes of signs introduce in the product so that $f(x)$ will have at least p changes of signs.

Hence, the number of changes of signs in $f(x)$ always exceeds the number of positive roots of $f(x)=0$.

Again the number of negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$ and therefore the number of changes of signs in $f(-x)$ always exceed the number of negative roots of $f(x)=0$.

## 4.3 SOLUTION OF CUBIC AND BIQUARDATIC EQUATION

### 4.3.1 Cubic Equation

Cubic equation is an equation of third degree. The most general form of a cubic equation is $x^3+ax^2+bx+c=0$ and by increasing each root of the cubic by $\frac{a}{3}$ it can be transformed into simpler form $x^3+px+q=0$.

### 4.3.2 Cardan's method of solving a cubic equation

Let the general cubic equation be $a_0x^3+3a_1x^2+3a_2x+a_3=0$ ...(i)

After removing the second term and multiplying the roots by $a_0$, equation (i) reduces to $\text{Z}^3\text{+3HZ+G=0}$ .....(ii)

Where $\text{Z=a}_0x+a_1$, $\text{H=a}_0a_2-a_1^2$ and $\text{G=a}_0{}^2a_3-3a_0a_1a_2+2a_1^3$

Let $\text{Z=u+v}$

By cubing, we get

$$\text{Z}^3=(u+v)^3 \Rightarrow \text{Z}^3=u^3+3uv(u+v)+v^3=u^3+3uvZ+v^3$$
$$\Rightarrow Z^3-3uvZ-(u^3+v^3) \qquad \text{... (iii)}$$

Comparing the coefficient of like terms in equations (ii) and (iii), we get

$\text{-3uv = 3H} \Rightarrow \text{uv = -H} \Rightarrow \text{u}^3v^3=-\text{H}^3$ and $\text{u}^3+v^3=-\text{G}$

Quadratic equation having roots $\text{u}^3, \text{v}^3$ is $\text{t}^2\text{-(u}^3\text{+v}^3\text{)t+u}^3v^3=0$

$$\Rightarrow \text{t}^2\text{+Gt-H}^3=0 \qquad \text{.... (iv)}$$

$$\Rightarrow \text{t}=\frac{\text{-G}\pm\sqrt{\text{G}^2+4\text{H}^3}}{2}$$

Suppose $\text{u}^3=\dfrac{\text{-G+}\sqrt{\text{G}^2+4\text{H}^3}}{2}$ and $\text{v}^3=\dfrac{-\text{G -}\sqrt{\text{G}^2+4\text{H}^3}}{2}$

From above we get three values of u and v as $\text{u, u}\omega\text{, u}\omega^2$ and $\text{v, v}\omega\text{, v}\omega^2$ respectively, where $\omega, \omega^2$ are the imaginary cube roots of unity.

We get the values of $\text{Z=u+v}$ by choosing, the values of u and v such that their product uv= -H is real. Thus, we have $\text{u+v, u}\omega\text{+v}\omega^2\text{, u}\omega^2\text{+v}\omega$ as the three values of Z and we can get the corresponding values of x from the relation $\text{Z=a}_0x+a_1$.

**Nature of the Roots of the Cubic equation** $Z^3+3\text{HZ}+\text{G}=0$

From the above section, we have $u^3 = \dfrac{-G+\sqrt{G^2+4H^3}}{2}$ and $v^3 = \dfrac{-G-\sqrt{G^2+4H^3}}{2}$ and hence the three values of u and v are $u, u\omega, u\omega^2$ and $v, v\omega, v\omega^2$ respectively, where $\omega, \omega^2$ are the imaginary cube roots of unity.

We obtain the values of Z = u+ v, by choosing the values of u and v such that their product uv= -H is real.

Let $\alpha, \beta, \gamma$ be the roots of equation (ii), then

$\alpha$=u+v

$$\beta = u\omega + v\omega^2 = u\left(\frac{-1+i\sqrt{3}}{2}\right) + v\left(\frac{-1-i\sqrt{3}}{2}\right) = -\frac{1}{2}(u+v) + i\frac{\sqrt{3}}{2}(u-v)$$

and $$\gamma = u\omega^2 + v\omega = u\left(\frac{-1-i\sqrt{3}}{2}\right) + v\left(\frac{-1+i\sqrt{3}}{2}\right) = -\frac{1}{2}(u+v) - i\frac{\sqrt{3}}{2}(u-v)$$

Now four cases arise:

**Case I:** If $G^2+4H^3$ is positive, $G^2+4H^3>0$, then the roots of equation (iv) are real

$\Rightarrow u^3$ and $v^3$ are both real or u and v are both real.

$\Rightarrow \alpha$ is real while $\beta$ and $\gamma$ are conjugate complex numbers.

**Case II:** If $G^2+4H^3$ is negative, i.e. $G^2+4H^3<0$, then the roots of equation (iv) are conjugate complex.

$\Rightarrow u^3$ and $v^3$ are conjugate complex numbers.

$\Rightarrow$ u and v are conjugate complex numbers.

$\therefore$ (u + v) is real and (u - v) is purely imaginary.

$\Rightarrow \alpha$ , $\beta$ and $\gamma$ are real roots

**Case III:** If $G^2+4H^3$ is zero, i.e. $G^2+4H^3=0$, then the roots of equation (iv) are equal

$\therefore u^3 = v^3 \Rightarrow u = v$

$\therefore$ All the roots are real while $\beta$ and $\gamma$ are equal roots.

**Case IV:** If G = H= 0, then equation (ii) becomes $Z^3 = 0 \Rightarrow \alpha = \beta = \gamma$

Hence, all the roots are equal.

**The irreducible case of Cardan's method**

If $G^2+4H^3$ is negative, i.e. $G^2+4H^3<0$ , then the roots of equation $t^3+Gt-H^3=0$ are conjugate complex.

$\Rightarrow u^3$ and $v^3$ are conjugate complex numbers.

$\Rightarrow$ u and v are conjugate complex numbers.

In this case, we use De-Moivre's theorem to find the cube roots of complex numbers.

Let $u^3 = a+ib$ and $v^3 = a-ib$ By putting $a=r\cos\theta$ and $b=r\sin\theta$ so that $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\dfrac{b}{a}\right)$

we get, $u^3 = r(\cos\theta + i\sin\theta) = r[\cos(\theta+2n\pi)+i\sin(\theta+2n\pi)]$

$\Rightarrow u = r^{\frac{1}{3}}[\cos(\theta+2n\pi)+i\sin(\theta+2n\pi)]^{\frac{1}{3}}$

$= r^{\frac{1}{3}}\left[\cos\left(\frac{\theta+2n\pi}{3}\right)+i\sin\left(\frac{\theta+2n\pi}{3}\right)\right]$, where n=0, 1, 2

Similarly, $v=r^{\frac{1}{3}}\left[\cos\left(\frac{\theta+2n\pi}{3}\right)-i\sin\left(\frac{\theta+2n\pi}{3}\right)\right]$,where n=0, 1, 2

$\therefore Z=u+v=r^{\frac{1}{3}}\left[\cos\left(\frac{\theta+2n\pi}{3}\right)+i\sin\left(\frac{\theta+2n\pi}{3}\right)\right]+r^{\frac{1}{3}}\left[\cos\left(\frac{\theta+2n\pi}{3}\right)-i\sin\left(\frac{\theta+2n\pi}{3}\right)\right]=2r^{\frac{1}{3}}\cos\left(\frac{\theta+2n\pi}{3}\right)$ where n=0, 1, 2

Hence, the three values of Z are

$2r^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right), 2r^{\frac{1}{3}}\cos\left(\frac{\theta+2\pi}{3}\right), 2r^{\frac{1}{3}}\cos\left(\frac{\theta+4\pi}{3}\right)$, where $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$.

**Example:** Solve the equation $28x^3-9x^2+1=0$ by Cardan's method.

**Solution:** The given equation is $28x^3-9x^2+1=0$ ...(i)

Here the third term is missing. First of all we transform it into an equation in which the second term from the beginning is missing. This can be done by substituting $x=\frac{1}{y}$ in (i).

Putting $x=\frac{1}{y}$ in (i), we get

$$\frac{28}{y^3}-\frac{9}{y^2}+1=0 \Rightarrow y^3-9y+28=0 \qquad ...(ii)$$

Let $y=u+v$

$\therefore$ $y^3=(u+v)^3$

or $y^3=u^3+v^3+3uv(u+v)=u^3+v^3+3uvy$

$y^3-3uvy-(u^3+v^3)=0$ ...(iii)

Comparing the coefficients of like terms in (ii) and (iii), we get

$uv=3 \Rightarrow (uv)^3=(3)^3 \Rightarrow u^3v^3=27$

and $u^3+v^3=-28$

Thus $u^3$ and $v^3$ are the roots of the equation

$t^2-(u^3+v^3)t+u^3v^3=0$

i.e. $t^2-28t+27=0$ i.e., $(t+1)(t+27)=0 \Rightarrow t=-1$ or $-27$

Let $u^3=-1$ and $v^3=-3$

One real value of u is -1 and one real value of v is -3

$\therefore$ One real value of $y=u+v$ is -4

i.e., One real value of $x=\frac{1}{y}$ is $-\frac{1}{4}$

Thus $-\frac{1}{4}$ is a root of equation (i). By synthetic division, we have

| | | | | |
|---|---|---|---|---|
| $-\frac{1}{4}$ | 28 | -9 | 0 | 1 |
| | | -7 | 4 | -1 |
| | 28 | -16 | 4 | 0 |

$\therefore$ Depressed equation of (i) is $28x^2 - 16x + 4 = 0$

or $\quad 7x^2 - 4x + 1 = 0$

$$\therefore \quad x = \frac{4 \pm \sqrt{16-28}}{14} = \frac{4 \pm \sqrt{-12}}{14} = \frac{4 \pm 2\sqrt{3}i}{14} = \frac{2 \pm \sqrt{3}i}{7}$$

Hence the roots are $-\frac{1}{4}, \frac{2 \pm \sqrt{3}i}{7}$.

**Example:** Solve the equation $x^3 + 3x - 14 = 0$ by Cardan's method.

**Solution:** The given equation $x^3 + 3x - 14 = 0$ ...(i)

Let $\quad x=u+v$

$\therefore \quad x^3=(u+v)^3$

or $\quad x^3=u^3+v^3+3uv(u+v)=u^3+v^3+3uvx$

i.e. $\quad x^3 - 3uvx - (u^3+v^3) = 0$ ...(iii)

Comparing the coefficients of like terms in (ii) and (iii), we get

$uv=-1 \quad \Rightarrow \quad (uv)^3 = (-1)^3 \Rightarrow u^3v^3=-1$

and $\quad u^3+v^3 = 14$

Thus $u^3$ and $v^3$ are the roots of the equation

$$t^2 - (u^3+v^3)t+u^3v^3 = 0$$

i.e. $\quad t^2 - 14t-1 = 0$

Solving for t, we get $t = \frac{14 \pm \sqrt{196+4}}{2} = \frac{14 \pm 10\sqrt{2}}{2} = \frac{14 \pm 10\sqrt{2}}{2} = 7 \pm 5\sqrt{2}$

Let $u^3 = 7 + 5\sqrt{2}$ and $v^3 = 7 - 5\sqrt{2}$

$\therefore$ u and v are of the form $p \pm q\sqrt{2}$

Let $\quad u^3 = 7 + 5\sqrt{2} = (p+q\sqrt{2})^3 \quad$ and $\quad v^3 = 7 - 5\sqrt{2} = (p-q\sqrt{2})^3$

Then $7 + 5\sqrt{2} = p^3 + 2q^3\sqrt{2} + 6pq^2 - 3p^2q\sqrt{2}$

and $7 - 5\sqrt{2} = p^3 - 2q^3\sqrt{2} + 6pq^2 - 3p^2q\sqrt{2}$

Adding and subtracting, we have

$$7 = p^3+6pq^2 = p(p^2+6q^2) \quad ...(iii)$$

and $\quad 5 = 2q^3 + 3p^2q = q(3p^2 + 2q^2)$ ...(iv)

From (iii) and (iv), it is clear that p is a factor of 7 and q is a factor of 5. By inspection we find that p=1, q=1 satisfy (iii) and (iv).

$\therefore \quad u = p + q\sqrt{2} = 1 + \sqrt{2}$ and $u = p - q\sqrt{2} = 1 - \sqrt{2}$ are real cube roots of $u^3$ and $v^3$ satisfying $uv = -1$

$\therefore \quad$ One root of (i) is given by $x = u + v = 2$

By synthetic division, we have

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 0 | 3 | -14 |
| | | 2 | 4 | 14 |
| | 1 | 2 | 7 | 0 |

$\therefore$ Depressed equation of (i) is $x^2+2x+7=0$

$$\therefore \qquad x=\frac{-2\pm\sqrt{4-28}}{2}=-1\pm i\sqrt{6}$$

Hence roots are -2, $-1\pm i\sqrt{6}$.

**Check Your Progress**

**1.** Solve the equation $x^3-12x-65=0$ by cardan's method.

Ans $5, \dfrac{-5\pm i3\sqrt{3}}{2}$

**4.4.1 Descarte's Solution of the Biquardatic Equation.**

Here we are going to find the roots of two different types of biquadratic equation using Descarte's method.

**Descarte's method for solving the Biquadratic equation**

$x^4+qx^2+rx+s=0$

The given biquadratic equation is $x^4+qx^2+rx+s=0$ ... (i)

Let the equation (i) can be split up two quadratic factors

$x^2+lx+m$ and $x^2-lx+n$

$\therefore x^4+qx^2+rx+s=(x^2+lx+m)(x^2-lx+n)$ ... (ii)

By comparing the coefficients of like terms in equation (ii),, we get

$m+n-l^2=q \Rightarrow n+m=l^2+q$ ... (iii)

$\ln - lm=r \Rightarrow n-m=\frac{r}{l}$ ... (iv)

and $mn=s$ ... (v)

By using, the identity $(n+m)^2-(n-m)^2=4mn$ we eliminate m and n.

From equation (iii), (iv) and (v), we get

$$(l^2+q)^2-\frac{r^2}{l^2}=4s \Rightarrow \left(l^4+2l^2q+q^2\right)-\frac{r^2}{l^2}=4s$$

$$\Rightarrow \left(l^6+2l^4q+l^2q^2\right)-r^2=4l^2s \Rightarrow l^6+2l^4q+l^2(q^2-4s)-r^2=0$$

Which is cubic equation in $l^2$. By trial method, $l^2$ and thus $l$ can be determined and then the values of m and n can be found from equation (iii) and (iv).

Hence, L.H.S of the equation (i) reduces to two quadratic factors $x^2+lx+m$ and $x^2-lx+n$. When these factors are equated to zero, we obtain four values of x, these four values of x are the roots of the given biquadratic equation.

**Descarte's method for solving the biquadratic equation**

$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$

Let the biquadratic equation be $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ ... (i)

To transform the given equation into another equation in which the second term is missing and the coefficient of the leading term is unity, we put $Z=a_0x+a_1$ in equation (i).

$Z^4+6HZ^2+4GZ+(a_0{}^2I-3H^2)=0$ ... (ii)

where, $H=a_0a_2-a_1^2$, $G=a_0{}^2a_3-3a_0a_1a_2+2a_1^3$ and $I=a_0a_4-4a_1a_3+3a_2^2$

Let equation (i) can be split up into two quadratic factors,

$Z^2+pZ+q$ and $Z^2-pZ+q'$

$\therefore Z^4+6HZ^2+4GZ+(a_0{}^2I-3H^2)=(Z^2+pZ+q)(Z^2-pZ+q')$ ... (iii)

Compairing the coefficients of like terms in equation (iii), we get

$q+q'-p^2=6H \Rightarrow q+q'=p^2+6H$ ... (iv)

$pq'-pq=4G \Rightarrow q'-q=\dfrac{4G}{p}$ ... (v)

and $qq'=a_0{}^2I-3H^2$ ...(vi)

By using the identity $(q+q')^2-(q-q')^2=4qq'$ we eliminate q and $q'$. From (iv), (v) and (vi), we get

$$(p^2+6H)^2-\left(\frac{4G}{p}\right)^2=4(a_0{}^2I-3H^2)$$

$$\Rightarrow (p^4+12p^2H+36H^2)-\frac{16G^2}{p^2}=4(a_0{}^2I-3H^2)$$

$$\Rightarrow p^6+12p^4H+36p^2H^2-16G^2=4p^2(a_0{}^2I-3H^2)$$

$$\Rightarrow p^6+12p^4H+4p^2(12H^2-a_0{}^2I)-16G^2=0$$

which is a cubic equation in $p^2$. Solving it by trial method, $p^2$ and p can be determined.

And then $q$ and $q'$ can be found from equation (iv) and (v).

Thus, L.H.S of the equation (iii) reduces to product of two quadratic factors. When these factors are equated to zero, we get four values of Z, which are the roots of the equation (ii) from the relation $Z=a_0x+a_1$, four values of x can be obtained.

Note: The identity

$Z^4+6HZ^2+4GZ+(a_0{}^2I)-3H^2)=(Z^2+pZ+q)(Z^2-pZ+q')$

is called Descarte's Resolvant.

**Example:** Apply Descarte's method to solve the equation $x^4-10x^3+35x^2-50x+24=0$.

Solution: The given equation is $x^4-10x^3+35x^2-50x+24=0$ ...(i)

To remove the second term, we diminish the roots by h where $h = -\frac{a_1}{na_0} = \frac{10}{4} = \frac{5}{2}$

and then multiplying the roots of (i) by 2, we get

$y^4 - 10.2x^3 + 35.2^2 x^2 - 50.2^3 x + 24.2^4 = 0$

$\Rightarrow y^4 - 20x^3 + 140x^2 - 400x + 384 = 0$ ...(ii)

where $y = 2x$

Now diminishing the roots of equation (ii) by 5 using synthetic division, we have

| | | | | | |
|---|---|---|---|---|---|
| 5 | 1 | -20 | 140 | -400 | 384 |
| | | 5 | -75 | 325 | -375 |
| | 1 | -15 | 65 | -75 | 9 |
| | | 5 | -50 | 75 | |
| | 1 | -10 | 15 | 0 | |
| | | 5 | -25 | | |
| | 1 | -5 | -10 | | |
| | | 5 | | | |
| | 1 | 0 | | | |
| | 1 | | | | |

$\therefore$ The transformed equation is $Z^4 - 10Z^2 + 9 = 0$ .... (iii)

where $Z = y - 5 = 2x - 5 \Rightarrow x = \frac{Z+5}{2}$ ...(iv)

From equation (iii), we have

$(Z^2 - 1)(Z^2 - 9) = 0 \Rightarrow Z = \pm 1, \pm 3$

Then from (iv), $x = \frac{1+5}{2}, \frac{-1+5}{2}, \frac{3+5}{2}, \frac{-3+5}{2}$ i.e.3,2,4,1

Thus, the required roots are 1,2,3 and 44.

**Check Your Progress**

**1.** Apply Descarte's method to solve the equation $x^4 - 3x^2 - 42x - 40 = 0$.

Ans. $\frac{-3 \pm i\sqrt{31}}{2}$, 4 and -1.

### 4.4.2 Ferrari Method of Solving a Biquadratic Equation

We will solve the biquadratic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ by splitting it into product of two quadratic factors by Ferrari's method. This is done by expressing the given biquadratic equation as the difference of two perfect squares.

The given biquadratic equation is $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ ...(i)

$\Rightarrow a_0x^4 + 4a_1x^3 = -6a_2x^2 - 4a_3x - a_4$

$\Rightarrow a_0{}^2x^4 + 4a_0a_1x^3 = -6a_0a_2x^2 - 4a_0a_3x - a_0a_4$ [Multiplying both sides by $a_0$]

Now, to make L.H.S. a perfect square, we add $4{a_1}^2x^2$ on both sides

$${a_0}^2x^4 + 4a_0a_1x^3 + 4{a_1}^2x^2 = 4{a_1}^2x^2 - 6a_0a_2x^2 - 4a_0a_3x - a_0a_4$$

$$\Rightarrow (a_0x^2 + 2a_1x)^2 = 4{a_1}^2x^2 - 6a_0a_2x^2 - 4a_0a_3x - a_0a_4 \qquad \text{... (ii)}$$

The above equation can be written as

$(a_0x^2 + 2a_1x + \lambda)^2 = 4{a_1}^2x^2 - 6a_0a_2x^2 - 4a_0a_3x - a_0a_4 + 2(a_0x^2 + 2a_1x)\lambda + \lambda^2$, where $\lambda$ is an arbitrary constant.

$$\Rightarrow (a_0x^2 + 2a_1x + \lambda)^2 = (4{a_1}^2 - 6a_0a_2 + 2a_0\lambda)x^2 + 4(a_1\lambda - a_0a_3)x + (\lambda^2 - a_0a_4) \qquad \text{....(iii)}$$

We choose $\lambda$ in such a way that R.H.S. becomes a perfect square of a linear expression in x. This is possible if and only if its discriminant is zero.

i.e. $$[4(a_1\lambda - a_0a_3)]^2 - 4(4{a_1}^2 - 6a_0a_2 + 2a_0\lambda)(\lambda^2 - a_0a_4) = 0$$
$$\Leftrightarrow \lambda^3 - 3a_2\lambda^2 + (4a_1a_3 - a_0a_4)\lambda + (3a_0a_2a_4 - 2{a_1}^2a_4 - 2a_0{a_3}^2) = 0$$

which is a cubic in $\lambda$ and is called Resolvent Cubic. Find the value of $\lambda$ which satisfies this cubic. Suppose the R.H.S of equation (iii) is a perfect square of linear expression px+q.

Thus, equation (iii) becomes,

$$(a_0x^2 + 2a_1x + \lambda)^2 = (px + q)^2$$

$$\Rightarrow (a_0x^2 + 2a_1x + \lambda)^2 - (px + q)^2 = 0$$

$$\Rightarrow \left[(a_0x^2 + 2a_1x + \lambda) - (px + q)\right]\left[(a_0x^2 + 2a_1x + \lambda) + (px + q)\right] = 0$$

$$\Rightarrow \left[a_0x^2 + (2a_1 - p)x + (\lambda - q)\right]\left[a_0x^2 + (2a_1 + p)x + (\lambda + q)\right] = 0$$

Thus, the L.H.S. of equation (iii) is written as the product of two quadratic factors $\left[a_0x^2 + (2a_1 - p)x + (\lambda - q)\right]$ and $\left[a_0x^2 + (2a_1 + p)x + (\lambda + q)\right]$ which when equated to zero gives four values of x, which are the roots of the equation (i).

Note: (i) To make the coefficient of $x^4$ unity, we multiply the roots of given equation by a suitable constant.

(ii) We can obtain roots of resolvent cubic by inspection.

**Example:** Solve the equation $2x^4 + 6x^3 - 3x^2 + 2 = 0$ by Ferrari's method.

Solution: The given equation is $2x^4 + 6x^3 - 3x^2 + 2 = 0$ ... (i)

In equation (i), the coefficient of $x^4$ is 2. So to make the coefficient of $x^4$ unity, we multiply roots of equation (i) by 2.

$$2^0.2y^4 + 2^1.6y^3 - 2^2.3y^2 + 2^3.2 = 0$$

$$\Rightarrow y^4 + 6y^3 - 6y^2 + 16 = 0 \qquad \text{... (ii)}$$

$$\Rightarrow y^4 + 6y^3 = 6y^2 - 16 \Rightarrow y^4 + 6y^3 + 9y^2 = 9y^2 + 6y^2 - 16$$

$$\Rightarrow (y^2 + 3y)^2 = 15y^2 - 16 \qquad \text{.... (iii)}$$

Now, $\left[(y^2 + 3y) + \lambda\right]^2 = (y^2 + 3y)^2 + 2(y^2 + 3y)\lambda + \lambda^2$

$= 15y^2 - 16 + 2(y^2 + 3y)\lambda + \lambda^2$ [Using (iii)]

$$=(15+2\lambda)y^2+6y\lambda+(\lambda^2-16) \qquad \text{...(iv)}$$

We choose $\lambda$ in such a way that R.H.S. of (iv) is a perfect square. This is possible if and only if discriminant of R.H.S. =0

$(6\lambda)^2-4(15+2\lambda)(\lambda^2-16)=0$

$9\lambda^2-(15+2\lambda)(\lambda^2-16)=0 \Rightarrow \lambda^3+3\lambda^2-16\lambda-120=0$

By inspection, $\lambda=5$ is a root of above equation.

By putting $\lambda=5$, equation (iv) becomes

$\left[y^2+3y+5\right]^2=25y^2+30y+9 \Rightarrow \left(y^2+3y+5\right)^2=\left(5y+3\right)^2$

$\Rightarrow \left(y^2+3y+5\right)^2-\left(5y+3\right)^2=0$

$\Rightarrow \left[\left(y^2+3y+5\right)-\left(5y+3\right)\right]\left[\left(y^2+3y+5\right)+\left(5y+3\right)\right]=0$

$\Rightarrow (y^2-2y+2)(y^2+8y+8)=0$

$\therefore$ Either $y^2-2y+2=0 \Rightarrow y=\dfrac{2\pm\sqrt{4-8}}{2}=\dfrac{2\pm 2i}{2}=1\pm i$

or $y^2+8y+8=0 \Rightarrow y=\dfrac{-8\pm\sqrt{64-32}}{2}=\dfrac{-8\pm 4\sqrt{2}}{2}=-4\pm 2\sqrt{2}$

which are the roots of equation (ii).

Dividing each root by 2, root of given equation (i) are $\dfrac{1\pm i}{2}$ and $-2\pm\sqrt{2}$.

**Check Your Progress**

1. Solve $x^4+2x^3-7x^2-8x+12=0$ by Ferrari's method.

Ans. -3, -2, 1, 2.

## 4.5 SUMMARY

- A continuation or permanence of sign is said to occur in a polynomial $f(x)$, whose terms are arranged in descending power of x, if the two successive terms have the same sign.
- The number of negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$ and therefore the number of changes of signs in $f(x)$ always exceed the number of negative roots of $f(x)$.
- We will reduce the cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$ to the form in which second term is missing and the coefficient of the leading term is unity, all other coefficients being integers.
- We will reduce the cubic $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ to the form in which second term is missing and the coefficient of the leading term is unity, all other coefficients being integers.

## 4.6 KEY TERMS

- **Real Polynomial:** A polynomial $f(x)=a_0x^n+a_1x^{n-1}+...+a_n$ is said to be real polynomial If all the coefficients $a_0,a_1,...,a_n$ in $f(x)$ are real.
- **Zero Polynomial:** If all the coefficients $a_0,a_1,...,a_n$ in $f(x)$ are zero then polynomial is said to be **zero polynomial**.

- **Complete and Incomplete equation:** A General Equation of degree n is said to be complete if it contains all power of the variable x from 0 to n. For example $a_0x^3 + a_1x^2 - a_3x + a_4 = 0$ is **complete equation** of third degree.
  If any of the powers of the variable are missing from an equation of degree n is called **incomplete equations**. For example $a_0x^3 + a_1x^2 + a_4 = 0$ is **incomplete equation** of third degree.
- **Root of an Equation:** The value of x for which $f(x)$ vanishes is called root of equation $f(x) = 0$. For example if $f(h) = 0$, then h is called root of the equation $f(x) = 0$.
- **Continuation or Permanence of sign:** A continuation or permanence of sign is said to occur in a polynomial $f(x)$, whose terms are arranged in descending power of x, if the two successive terms have the same sign.

## 4.7 QUESTIONS AND EXERCISES

1. Solve the equation $28x^3 - 9x^2 + 1 = 0$ by Cardan's method.
2. Show that the roots of the cubic equation $x^3 - 12x^2 + 8 = 0$ are
   $4\cos\frac{2\pi}{9}, 4\cos\frac{4\pi}{9}, 4\cos\frac{8\pi}{9}$
3. Solve the equation by expressing them as product of two quadratic factors
   (i) $x^4 - 5x^2 - 6x - 5 = 0$ (ii) $x^4 - 8x^2 - 24x + 7 = 0$
4. Solve the given equation by Descarte's method $2x^4 + 6x^3 - 3x^2 + 2 = 0$.
5. Solve the equation $x^4 - 15x^2 + 20x - 6 = 0$ by the method of resolution into quadratic factors.
6. Solve the equation by using Ferrari's method $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

## 4.8 FURTHER READING

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